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Spitzer identity, Wiener-Hopf factorization and pricing of discretely monitored exotic options

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ABSTRACT

The Wiener-Hopf factorization of a complex function arises in a variety of fields in applied mathematics such as probability, finance, insurance, queuing theory, radio engineering and fluid mechanics. The factorization fully characterizes the distribution of functionals of a random walk or a Lévy process, such as the maximum, the minimum and hitting times. Here we propose a constructive procedure for the computation of the Wiener-Hopf factors, valid for both single and double barriers, based on the combined use of the Hilbert and the z-transform. The numerical implementation can be simply performed via the fast Fourier transform and the Euler summation. Given that the information in the Wiener-Hopf factors is strictly related to the distributions of the first passage times, as a concrete application in mathematical finance we consider the pricing of discretely monitored exotic options, such as lookback and barrier options, when the underlying price evolves according to an exponential Lévy process. We show that the computational cost of our procedure is independent of the number of monitoring dates and the error decays exponentially with the number of grid points.

1. Introduction

This paper provides a new procedure to determine the finite-time distribution of the discrete extrema and of the hitting times of one or two barriers for a process with independent and identically distributed increments, such as a Lévy process. Spitzer (1956) provided a closed formula for the z-transform (or moment generating function or discrete Laplace transform) of the characteristic function of the extrema of a random walk observed on a set of discrete dates. Up to now the concrete application of the Spitzer identity has been difficult because it requires the Wiener-Hopf (WH) factorization of a function defined in the complex plane, a mathematical problem that concerns a variety of fields in applied mathematics. Indeed, this factorization cannot be achieved analytically except in few cases, or its computation turns out to be very demanding requiring the numerical evaluation of a multidimensional integral in the complex plane. In addition, with regard to a general Lévy process, little is known for the two-barriers case.

The key contributions of our paper are the following. First of all, we provide a constructive procedure for performing the WH factorization. More precisely, we express the WH factors arising in the Spitzer identity in terms of the Plemelj–Sokhotsky relations, which allow us to compute the WH factors through the Hilbert transform. The latter is then approximated via a sinc function expansion (Stenger, 1993), which guarantees an exponential decay of the approximation error on the number of grid points.

Moreover, our methodology can deal with both a single and a double barrier. The solution in the second case is of interest in itself because it solves a long-standing problem related to an efficient computation of the WH factors in the presence of two barriers. The double-barrier case did not admit a simple feasible solution up to now, except under few special assumptions on the structure of the Lévy process. One has to solve two coupled integral equations, which can be achieved by factorizing a 2×2 matrix of functions, but a general analytical method for this more difficult problem has not been found yet (Jones, 1991). Here, as the second main contribution of the paper, we constructively propose a fixed-point algorithm based on an extension of the single-barrier case that achieves a fast convergence.

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As a concrete application, we contribute to the mathematical finance literature related to the pricing of exotic options, such as barrier and lookback. Pricing derivatives, especially exotic options, is a challenging problem in the operations research literature (to cite a few, see Cai, Chen, & Wan, 2009; Date & Islyaev, 2015; Dingeç & Hörmann, 2012; Feng & Linetsky, 2008b; Giesecke & Smelov, 2013; Jin, Li, Tan, & Wu, 2013; Sesana, Marazzina, & Fusai, 2014; Wang & Tan, 2013). The application of transform techniques in mathematical finance is rather recent. The first and most important contributions are probably the articles by Heston (1993) and Carr and Madan (1999), where the authors show how to price European options with non-Gaussian models exploiting the Fourier transform. Similar techniques were developed later for path-dependent derivatives (e.g. Cai et al., 2009; Feng & Linetsky, 2008a; Green, Fusai, & Abrahams, 2010). Our paper provides a unified framework and a fast operational method for pricing barrier and lookback (or hindsight) options when the underlying asset evolves as an exponential Lévy process. In addition, the monitoring condition, e.g., the event that the underlying asset value falls below a given barrier for a down-and-out barrier option, is assumed to be controlled at discrete time intervals. Our procedure, based on the new WH factorization method, has a computational cost independent of the number of monitoring dates. This is possible because the inversion of the discrete Laplace transform is performed via the Euler acceleration, which bounds from above the number of WH factorizations to be computed. Moreover, at least with regard to single-barrier and lookback options, the method provides exponential order of convergence due to the fact that the factorization is performed remaining in the complex plane. The existing pricing methods are based on the backward recursive formula (e.g. Fusai, Longo, Marena, & Recchioni, 2009; Fusai, Marazzina, Marena, & Ng, 2012; Fusai & Recchioni, 2007; Jackson, Jaimungal, & Surkov, 2008; Lord, Fang, Bervoets, & Oosterlee, 2008), and on exploiting the convolution structure of the transition density of the Lévy process by performing the computations efficiently and fast using the FFT, which leads to a CPU time that grows as $\mathcal{O}(M\log M)$, where M is the number of grid points. However, all the above cited methods are characterized by a polynomial decay of the error with M. This order of accuracy is related to the fact that the backward procedure for barrier options involves a convolution, that can be computed in the complex plane, and a projection, which is applied in the real plane, to take into account the presence of the barrier. A noticeable exception was presented by Feng and Linetsky (2008a, 2009), who reformulated the backward procedure for barrier and lookback options in terms of the Hilbert transform, so that all steps are performed in the complex plane. Computing the Hilbert transform with a sinc function expansion, they achieved an exponential decay of the error. However, the computational cost of all these methods, including the one by Feng and Linetsky, increases linearly with the number of monitoring dates.

Finally, the factorization procedure introduced here is quite general and can also be applied, without any additional complication, to continuously-monitored contracts. Even the best available method listed above, i.e., that by Feng and Linetsky, does not have this feature.

Even if the Spitzer identity has already been used in option pricing (e.g. Atkinson & Fusai, 2007; Borovkov & Novikov, 2002; Green et al., 2010; Lewis & Mordecki, 2008) and the present paper is mainly focused on this kind of applications, our method goes well beyond option pricing and opens up the way to a more extensive use of the Spitzer identity and the WH factorization in a variety of non-financial fields; for physics, see a recent review by Bray, Majumdar, and Schehr (2013). In this regard we would like to mention the applicability to queuing theory due to the strict connection between random walks and queues, see Lindley (1952) for pioneering contributions as well as Cohen (1975), Prabhu (1974) and Asmussen (1987); 1998). Further applications include insurance (Gerber, Shiu, & Yang, 2013) and sequential testing (Siegmund, 1985). Finally, the WH factorization arises in many branches of engineering, mathematical physics and applied mathematics. This is testified by the thousands of papers published on the subject since its conception. A review of the different applications is given by Lawrie and Abrahams (2007).

The structure of the paper is the following. Section 2 introduces the Spitzer identity and its relationship with the WH factorization, proposing, via the interpretation of the Plemelj–Sokhotsky relations as Hilbert transforms, a new operational method to perform the factorization and therefore to compute the distributions of the minimum and the maximum of a Lévy process, as well as the joint distributions of the process at maturity and of its minimum or maximum over the whole time interval. Section 3 shows how the proposed general methodology can be implemented efficiently and accurately computing the Hilbert transform via a sinc expansion; we also discuss the inversion of the z-transform and its acceleration through the Euler summation rule to make the computational cost independent of the number of monitoring dates. Section 4 deals with the pricing problem for lookback and barrier options, describing how our procedure is fast as well as accurate. This is validated numerically in Section 5 with a variety of numerical experiments.

2. Spitzer identity and Wiener-Hopf factorization

We consider a Lévy process $X(t)$, i.e., a stochastic process with $X(0)=0$ and independent and identically distributed increments. The Lévy–Khincine formula states that the characteristic function of the process is given by $\Psi(\xi,t)=\mathbb{E}[e^{i\xi X(t)}]=e^{\psi(\xi)t}$, where ψ is the characteristic exponent of the process,

$$\psi(\xi)=ia\xi-\frac{1}{2}\sigma^2\xi^2+\int_{\mathbb{R}}(e^{i\xi\eta}-1-i\xi\eta\mathbf{1}_{|\eta|<1})\nu(d\eta). \tag{1}$$

The Lévy–Khincine triplet (a,σ,ν) fully defines the Lévy process $X(t)$.

In several applications in queueing theory, insurance and financial mathematics, the key point is the determination of the law of the extrema of the Lévy process observed on an equally-spaced grid $X_n=X(n\Delta),\quad n=0,\dots,N$, where $\Delta>0$ is the time step, i.e., the distance between two consecutive monitoring dates, which is assumed constant. We define the processes of the maximum M_N and of the minimum m_N up to the Nth monitoring date as

$$M_N=\max_{n=0,\dots,N}X_n \quad\text{and}\quad m_N=\min_{n=0,\dots,N}X_n. \tag{2}$$

To distinguish the present case, where the above processes, albeit evolving in continuous time, are recorded only at discrete times, the terminology discrete versus continuous monitoring is used.

In particular, besides the distribution $P_X(x,N)$ of the Lévy process at maturity $T=N\Delta$, we will need the distributions $P_m(x,N)$ of the minimum and $P_M(x,N)$ of the maximum over the whole set $\{n=0,\dots,N\}$, as well as the joint distributions $P_{X,m}(x,N)$ or $P_{X,M}(x,N)$ of the process at maturity and of its minimum or maximum over the interval with respect to a lower (upper) barrier l (u), and the joint distribution of the triplet (X_N,m_N,M_N), $P_{X,m,M}(x,N)$. These distributions are defined as

$$dP_X(x,N)=p_X(x,N)dx=\mathbb{P}[X_N\in[x,x+dx)] \tag{3}$$

$$dP_m(x,N)=p_m(x,N)dx=\mathbb{P}[m_N\in[x,x+dx)] \tag{4}$$

$$dP_M(x,N)=p_M(x,N)dx=\mathbb{P}[M_N\in[x,x+dx)] \tag{5}$$

$$dP_{X,m}(x,N)=p_{X,m}(x,N)dx=\mathbb{P}[X_N\in[x,x+dx),m_N>l] \tag{6}$$

$$dP_{X,M}(x,N)=p_{X,M}(x,N)dx=\mathbb{P}[X_N\in[x,x+dx),M_N<u] \tag{7}$$

$$\begin{aligned}dP_{X,m,M}(x,N)&=p_{X,m,M}(x,N)dx\\&=\mathbb{P}[X_N\in[x,x+dx),m_N>l,M_N<u].\end{aligned} \tag{8}$$