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New shock models based on the generalized Polya process

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ABSTRACT

Various shock models have been extensively studied in the literature, mostly under the assumption of the Poisson process of shocks. In the current paper, we study shock models under the generalized Polya process (GPP) of shocks, which has been recently introduced and characterized in the literature (see Konno (2010) and Cha, 2014). Distinct from the widely used nonhomogeneous Poisson process, the important feature of this process is the dependence of its stochastic intensity on the number of previous shocks. We consider the extreme shock model, where each shock is catastrophic for a system with probability $p(t)$ and is harmless with the complementary probability $q(t) = 1 - p(t)$. The corresponding survival and the failure rate functions are derived and analyzed. These results can be used in various applications including engineering, survival analysis, finance, biology and so forth. The cumulative shock model, where each shock results in the increment of wear and a system's failure occurs when the accumulated wear reaches some boundary is also considered. A new general concept describing the dependent increments property of a stochastic process is suggested and discussed with respect to the GPP.

1. Introduction

Various shock models have been extensively studied and applied to various topics in reliability (see, e.g., Cha & Finkelstein (2010), Chakravarthy (2012), Frostig & Kenzin (2009), Huynh, Castro & Barros (2012), Montoro-Cazorla and Pérez-Ocón (2011)). In the literature, most of the shock models have been developed under the assumption of the Poisson process of shocks (see, e.g., Finkelstein & Cha, 2013; Nakagawa, 2007 and references therein). The nonhomogeneous Poisson process (NHPP), due to its relative simplicity, is the most popular point process in numerous engineering and biological applications and, specifically, in shock modeling. For instance, the famous Strehler–Mildvan model of human mortality (Strehler & Mildvan, 1960) considers the Poisson process of shocks (demands for energy) affecting an organism, which eventually results in 'justification' of the Gompertz law of human mortality. Various Poisson shock models are usually mathematically tractable and allow for rather simple and compact expressions for the probabilities of interest (see, e.g., Al-Hameed & Proschan, 1973; Cha & Finkelstein, 2009, 2011; Cha & Mi, 2007; Esary, Marshal, & Proschan, 1973 to name a few). It is worth mentioning that shock models governed by the renewal processes, which have a simple probabilistic nature, are already more cumbersome and approximations and numerical methods should be used (Finkelstein, 2003; Kalashnikov, 1997).

It is well known that the NHPP possesses the property of independent increments that along with other properties enables reasonably simple probabilistic reasoning. However, the assumption of independence of increments, in fact, can be too restrictive to describe most of the real life problems. For instance, in various shock models, a system's susceptibility to shocks increases with the number of shocks experienced previously. Thus a minor or even a negligible shock that had occurred during the initial lifetime period can become harmful and even catastrophic with time.

Recently, a new counting process, the 'generalized Polya process (GPP)', has been studied and characterized in Cha (2014) (see also Konno (2010) for the formal definition of this process). In the current paper, we consider the GPP as our baseline process for the corresponding shock models. This process is defined via its stochastic intensity that takes into account the number of previous shocks and, in this way, it creates a rather simple but effective model which depends on the history. The GPP defined in this way possesses a positive dependence property which means that the susceptibility of the event occurrence in an infinitesimal interval of time increases as the number of events in the previous interval increases. This property is definitely relevant to various applications, where the assumption of the independent increments of the NHPP was often used just for simplicity.

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We focus on the extreme shock model for the case when the shock process follows the GPP. Each shock in this model is considered to be critical (catastrophic) to a system with the time dependent probability $p(t)$ and harmless with a complementary probability $q(t) = 1 - p(t)$ (independently of 'everything else'). The survival probability of a system (without critical shocks) is of interest. This is a classical reliability setting with a well-known solution for the NHPP process of shocks (Beichelt & Fischer, 1980; Block, Borges & Savits, 1985; Finkelstein, 2008). However, the problem is more complex for the GPP case and the corresponding solution should be carefully derived, which is the main result of Section 2. In addition, in this section, some generalizations to the case of the cumulative shock model are briefly discussed when a failure of a system occurs if the accumulated wear (damage) reaches some given threshold level. The emphasis of the paper is on the analysis of the obtained survival function and the corresponding failure rate. Therefore, in Section 3, we derive the relation for the failure rate using a different (point process) approach that allows for this important analysis. Furthermore, a new general concept describing the dependent increments property of a stochastic process is suggested and discussed with respect to the GPP. Finally, concluding remarks are given in Section 4.

The new meaningful analysis performed in this paper will constitute a basis for further development of the model for various applications in engineering, survival analysis, finance, biology and so forth.

2. Extreme shock model

A new counting process, called the 'Generalized Polya Process' (GPP) has been recently described based on the notion of stochastic intensity and its properties have been studied in detail in Cha (2014) (see also Konno (2010)) . As its definition was based on the corresponding stochastic intensity, let us first briefly discuss this notion in a way suitable for further presentation.

Let $\{N(t),\ t \ge 0\}$ be a *simple* (or *orderly*) point process and $H_{t-} \equiv \{N(u),\ 0 \le u < t\}$ be the history (internal filtration) of the process in $[0, t)$, i.e., the set of all point events in $[0, t)$. Observe that H_{t-} can equivalently be defined in terms of $N(t-)$ and the sequential arrival points of the events $T_0 \equiv 0 \le T_1 \le T_2 \le \ldots \le T_{N(t-)} < t$ in $[0, t)$, where T_i is the time from 0 until the arrival of the ith event in $[0, t)$. The point processes can be conveniently mathematically described by using the concept of the stochastic intensity (the intensity process) λ_t, $t \ge 0$ (Aven & Jensen, 1999, 2000). As discussed, e.g., in Cha and Finkelstein (2011), and Cha (2014), the stochastic intensity λ_t of an orderly point process $\{N(t),\ t \ge 0\}$ is defined as the following limit:

$$\lambda_t = \lim_{\Delta t \to 0} \frac{\Pr[N(t, t+\Delta t) = 1 | H_{t-}]}{\Delta t} = \lim_{\Delta t \to 0} \frac{E[N(t, t+\Delta t) | H_{t-}]}{\Delta t}, \tag{1}$$

where $N(t_1, t_2)$, $t_1 < t_2$, represents the number of events in $[t_1, t_2)$. The stochastic intensity defined in (1) has the following heuristic interpretation: $\lambda_t dt = E[dN(t)|H_{t-}]$ (Aven & Jensen, 1999), where λ_t reduces to the ordinary failure (hazard) rate of a random variable when $H_{t-} = \{N(t-) = 0\}$ meaning that there were no events in $[0, t)$. It is well known that for the NHPP with intensity function $\lambda(t)$, the stochastic intensity is deterministic and equal to $\lambda(t)$, whereas for the renewal process with the failure rate of the underlying distribution $\lambda(t)$, it is given by the following relationship $\lambda_t = \lambda(t - T_{N(t-)})$, $t \ge 0$, where $T_{N(t-)}$, as usual, denotes a random time of the last renewal in $[0, t)$ (see, e.g., Finkelstein & Cha (2013)).

Now we are ready to provide a formal definition of the GPP (Cha, 2014).

Definition 1. ***Generalized Polya Process (GPP)***

A counting process $\{N(t),\ t \ge 0\}$ is called the Generalized Polya Process (GPP) with the set of parameters $(\lambda(t), \alpha, \beta)$, $\alpha \ge 0$, $\beta > 0$, if

(i) $N(0) = 0$;
(ii) $\lambda_t = (\alpha N(t-) + \beta)\lambda(t)$.

As mentioned in Cha (2014), the GPP with $(\lambda(t), \alpha = 0, \beta = 1)$ reduces to the NHPP with the intensity function $\lambda(t)$ and, accordingly, the GPP can be understood as a generalized version of the NHPP. It should be noted that recently a similar model was studied in Asfaw and Linqvist (2015) (see also Le Gat, 2014; Babykina & Couallier, 2014). However, the focus of these papers was different, mostly considering the corresponding issues of statistical inference and related frailty modeling. It is also clear that, by suitable reparameterization, we can always transform the model to the case when $\beta = 1$. However for the sake of further presentation (e.g., the 'restarting property' based on Definition 1 is used in this section for discussing the residual lifetime of the system and for characterizing the dependence structure of the GPP in Section 3), we keep the above setting as in the original paper by Cha (2014).

Consider now a system subject to the GPP of shocks with the set of parameters $(\lambda(t), \alpha, \beta)$. Let it be 'absolutely reliable' in the absence of shocks. As before, $T_0 \equiv 0 \le T_1 \le T_2 \le \ldots$ denote the sequential arrival times of the GPP. Assume that a shock that had occurred at time t results in the system's failure with probability $p(t)$ and is harmless to the system with probability $q(t) = 1 - p(t)$ independent of everything else. This shock model is usually called in the literature the 'extreme shock model'. First of all, within the frame work of the extreme shock model and keeping in mind various applications, we will be interested in the probability of survival of our system subject to the GPP. To derive the survival probability, we need the following supplementary results.

Lemma 1. *For the GPP with the set of parameters $(\lambda(t), \alpha, \beta)$, $\alpha > 0$, $\beta > 0$, the following properties hold:*

(i) *The distribution of $N(t)$ is given by*

$$P(N(t) = n) = \frac{\Gamma(\beta/\alpha + n)}{\Gamma(\beta/\alpha) n!} (1 - \exp\{-\alpha\Lambda(t)\})^n \times (\exp\{-\alpha\Lambda(t)\})^{\frac{\beta}{\alpha}},\ n = 0, 1, 2, \ldots$$

(ii) *The conditional joint distribution of $(T_1, T_2, \ldots, T_{N(t)})$ given that $N(t) = n$ is*

$$f_{(T_1, T_2, \ldots, T_{N(t)}|N(t))}(t_1, t_2, \ldots, t_n|n) = n! \prod_{i=1}^{n} \frac{\alpha\lambda(t_i)\exp\{\alpha\Lambda(t_i)\}}{\exp\{\alpha\Lambda(t)\} - 1},\ 0 \le t_1 \le t_2 \le \ldots \le t_n,$$

where $\Lambda(t) \equiv \int_0^t \lambda(u) du$.

For the proofs of (i) and (ii), see the proofs of Theorems 1 and 3 in Cha (2014), respectively.

It follows from Lemma 1 that

$$E[N(t)] = \frac{\beta}{\alpha}(\exp\{\alpha\Lambda(t)\} - 1).$$

Therefore, the intensity function of the GPP is defined as

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} P(N(t + \Delta t) - N(t) = 1) = \frac{d}{dt} E[N(t)] = \beta\lambda(t)\exp\{\alpha\Lambda(t)\}.$$

Thus even for the constant baseline function, $\lambda(t) = \lambda$, the intensity function is exponentially increasing which reflects the cumulative effect of the previous events on the probability of occurrence of an event at the current instant of time.