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Should European gamblers play lotto in the USA?

Vera Hofer *, Johannes Leitner

Department of Statistics and Operations Research, Karl Franzens University, Universitätsstrasse 15/E3, 8010 Graz, Austria

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ABSTRACT

Any jackpot building game is designed to have a negative expected return for the gambler, but it can be profitable under certain circumstances. Previous studies have shown that the purchase of a single ticket of US-American state lotteries is sometimes a gamble with a positive expected value. Lottery winnings are not taxed in Europe, which suggests that the profitability of European games may be even higher. We present an exact formula for the calculation of the expected value of a single lotto ticket and find European lottery drawings to be far less profitable for the gambler compared to the US-American lotto market. Those US lotteries that generate profitable drawings are not characterized by higher redistribution rates or by their specific rules, but by the purchasing behavior of the gamblers. These gamblers buy far fewer tickets (per capita) and they barely react to increasing jackpots, even though the jackpots are large enough to cause positive expected winnings.

1. Introduction

Lotteries provide a rich application for OR-specific research. In the general case they allow experimental or theoretical analyses of utility theories, see Bakir and Klutke (2011) or Pope et al. (2009). A particular type of lottery is the game lotto, where gamblers choose a combination of a out of b numbers. How many of these a numbers match randomly drawn a numbers in the lotto drawing determines the amount of winnings. Most studies on state-run lotto games analyse the game from the perspective of the provider. Examples include the optimal design of lotto games (Quiggin, 1991; Hartley and Lanot, 2003), or the optimal variation of the redistribution rate given the size of the rollover to maximize profits (Beenstock et al., 2000). In contrast, Droesbeke and Lorea (1982), and Jans and Degraeve (2008) investigated a gambler-specific question. They tried to solve the so-called *lotto problem*, i.e. identification of the minimum number of tickets such that there is at least one ticket with at least x matching numbers. The present paper also looks at the lotto game from the perspective of the gambler and tries to answer the question what conditions favor the occurrence of profitable drawings and whether the gambler should buy a single ticket or not.

Lotto is one of the least profitable gambles for a player because only a fraction $d \in [0, 1]$ of the revenue from ticket sales (at a ticket price P) is redistributed to the players. The expected return of a ticket depends on the amount of money in the pot, which consists of the rollover R from earlier drawings in which no ticket was a winner, and the number of additional n tickets other players buy, thus contributing $n \times d \times P$ to the total pot. The more tickets sold the larger the total pot $(R + n \times d \times P)$ becomes, but the probability of having to share the prize with other winning tickets also increases. Given a sufficiently high rollover, the purchase of a lotto ticket is therefore not necessarily an irrational decision. Cook and Clotfelter (1993), Matheson (2001) and Forrest et al. (2002) present approximate calculations of the expected return of a lottery ticket ex post. Earlier empirical analyses of lotto drawings mostly comprise very few drawings with high jackpots, e.g. Matheson (2001). In a large scale study, Grote and Matheson (2006) found positive expected values for single tickets in 1% of 18,252 US-American lotto drawings, reaching 5% of all drawings in certain states. The reasons why certain lotteries were profitable and others not, given comparable distribution rates and tax rates, were not addressed.

In US lotteries the state taxes the gamblers twice: First, on purchase of the ticket, by retaining $1 - d$ of the ticket sales, and secondly, by imposing income taxes on the winnings. We hypothesize that profitable situations occur more often in European lotteries because lottery winnings are free from income tax. Taking the specific characteristics of European lotto games into account we present an exact calculation of the expected ticket return and applying an ex ante criterion we empirically test the profitability of a single ticket. We find that contrary to our hypothesis, European lotto games are in fact very rarely profitable. We then discuss the reasons for this.

2. A typical European lotto game

The reward structure of a lotto game consists of T different winning tiers with fixed probabilities p_i, where $i = 1, \ldots, T$. The tier with the smallest winning probability typically contains the

* Corresponding author. Tel.: +43 316 380 7243; fax: +43 316 380 9560.
E-mail address: vera.hofer@uni-graz.at (V. Hofer).

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Table 1
Reward structure in the Austrian 'Lotto 6 aus 45'.

Tier	Correct numbers	Probability (p_i)	Reward of total pot (w_i) (%)
1	6	1:8,145,060	42
2	5 + 1	1:1,357,510	8
3	5	1:35,724	9
4	4	1:733	16
5	3	1:45	25

rollover R of earlier drawings where no ticket matched the combination of numbers drawn. In US-American lotto games the lower tiers pay fixed amounts, and only the highest tier contains the variable component. In European lotto games, usually the winning payments of all tiers depend on the total pot. In only very few states does the lowest tier contain a fixed reward.

For example, in the Austrian state lottery the player needs to choose 6 out of 45 possible numbers and 7 numbers are drawn twice weekly, where the seventh number is called a 'bonus' number used for a special combination of winning options, the so-called $5+1$ winning tier. An Austrian lotto ticket costs $P = 1$ Euro and contributes only $d = 43.65\%$ of total sales to the payout in the next drawing. Five tiers receive a fraction of the total pot as summarized in Table 1.[1]

Guessing the 6 standard numbers correctly has a probability of $p = 1/8,145,060$ and is rewarded with $w_1 = 42\%$ of the pot which consists of $n \times d \times P$ plus the rollover from earlier drawings R_1. If there is no winning ticket for tier i in drawing $t, R_i = n \times w_i \times d \times P$ is rolled onto the next drawing in $t+1$. Tickets qualifying for a winning tier i in $t+1$ are rewarded a fraction of the pot of drawing $t+1$ plus R_i. In the Austrian lotto $R_1 > 0$ occurs in about 50% of all drawings, $R_2 > 0$ in about 5% of the drawings. Rollovers in the other tiers have so far not occured. The jackpot advertised by the Austrian lotteries is an estimation of the ticket sales $\hat{n}$ in the next drawing and equals $R_1 + 0.4365 \times 0.42 \times \hat{n}$. This estimate is very accurate but it is not binding if the true n differs from the estimate $\hat{n}$.

3. The expected value of a single lottery ticket

For a simplified lotto game with only one winning tier, no rollover ($R = 0$), a ticket price P, a distribution rate d, a winning probability p, and a probability of loss $q = 1 - p$, the expected winnings for the buyer of one ticket are $E = 2Pd(p^2\frac{1}{2} + pq)$ if there is only one other ticket buyer, and $E = 3Pd(\frac{1}{3}p^3 + p^2q\frac{1}{2} + p^2q\frac{1}{2} + pq^2)$ if there are two other tickets. In the general case of n tickets the expected winnings are:

$$E = f(n) = Pnd\sum_{k=0}^{n-1} \underbrace{\frac{1}{n-k}\binom{n-1}{k}}_{=\frac{(n-1)!}{(n-k)(n-1-k)!k!}} p^{n-k}q^k$$

$$= Pd\sum_{k=0}^{n-1}\frac{n!}{(n-k)!k!}p^{n-k}q^k = Pd\underbrace{\sum\nolimits_{k=0}^{n-1}\binom{n}{k}p^{n-k}q^k}_{=1-q^n} \tag{1}$$

The sequence $f(n) = Pd(1-q^n)$ increases monotonically since $f(n+1) - f(n) = Pd(q^n - q^{n+1}) > 0$ and it converges to $\lim_{n\to\infty} Pd(1-q^n) \to Pd$. This simple game has two important characteristics. First, the expected winnings increase monotonically when more players buy tickets. Second, for the buyer of a single ticket the expected value of taking part in the game converges to the money value that he contributes to the pot. Therefore playing in a lotto game without a rollover cannot have a positive expected return.

In a game with only one winning tier and a rollover R from earlier drawings, the expected winnings E increase to

$$E = (Pnd + R)\sum_{k=0}^{n-1}\frac{1}{n-k}\binom{n-1}{k}p^{n-k}q^k = \left(Pd + \frac{R}{n}\right)(1-q^n).$$

The sequence of expected winnings has the same limit $\lim_{n\to\infty}(Pd + \frac{R}{n})(1-q^n) \to Pd$ for $R = 0$ and $R > 0$. The monotonicity of the sequence for $R > 0$ is not clear since

$$f(n+1) - f(n) = Pd\underbrace{(q^n - q^{n+1})}_{>0} + R\underbrace{\left(\frac{1}{n}q^n - \frac{1}{n+1}q^{n+1}\right)}_{>0} + R\underbrace{\left(\frac{1}{n+1} - \frac{1}{n}\right)}_{<0}$$

The calculation of expected winnings can be extended to the more realistic case of games with several winning tiers $i = 1, \ldots T$. Each is then funded with a certain share w_i of the total pot Pnd and with a potential rollover R_i. Adapting (1) to cover several tiers, the expected winnings are

$$E = f(n) = \sum_{i=1}^{T}\left(Pdw_i + \frac{R_i}{n}\right)(1-q_i^n) \tag{2}$$

The limit of (2) for $n \to \infty = Pd$. With increasing ticket sales, the rollover does not change the expected value of the game.

We want to know when the expected return of a ticket is positive, i.e. when $E - P > 0$. From the discussion of (1) we already know that increasing ticket sales, i.e. increasing n, cannot generate a situation with a positive expected return. A certain rollover is also required. To find this total rollover we rearrange (2) such that $E > P$:

$$\sum_{i=1}^{T}R_i(1-q_i^n) > Pn\left(1 - d\sum_{i=1}^{T}w_i(1-q_i^n)\right) \tag{3}$$

Returning now to the example of the Austrian lottery, we find that in most cases a rollover occurs in tier 1. The rollover R_1 that is required in the Austrian lotto to cause a positive expected value when only one ticket is bought and $R_2, \ldots, R_5 = 0$ is approximately €8,124,298. The required size of R_1 does not increase monotonically but has a local minimum. This can be seen from Fig. 1.

The sequence $f(n)$ has its minimum at total ticket sales of $n = 213,473$. Below the minimum size of the rollover

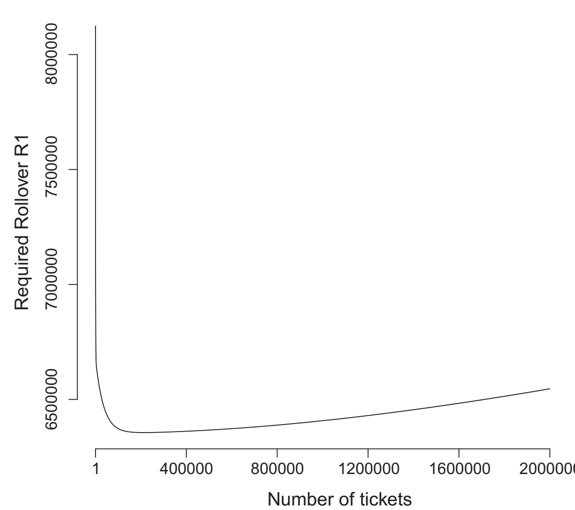

Fig. 1. Required size of the rollover to make the Austrian lotto game profitable.

[1] Table 1 presents the rules of the game as they were available from the website of the Austrian Lotteries www.win2day.at. The rules of the game were recently changed on the 8th of September 2010 to include a higher number of winning tiers, a minimum rollover R_1 of 1 million Euros and a higher ticket price of $P = 1.1$ Euros. These changes are not considered in this study.