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Pricing derivatives with counterparty risk and collateralization: A fixed point approach

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ABSTRACT

This paper studies a valuation framework for financial contracts subject to reference and counterparty default risks with collateralization requirement. We propose a fixed point approach to analyze the mark-to-market contract value with counterparty risk provision, and show that it is a unique bounded and continuous fixed point via contraction mapping. This leads us to develop an accurate iterative numerical scheme for valuation. Specifically, we solve a sequence of linear inhomogeneous PDEs, whose solutions converge to the fixed point price function. We apply our methodology to compute the bid and ask prices for both defaultable equity and fixed-income derivatives, and illustrate the non-trivial effects of counterparty risk, collateralization ratio and liquidation convention on the bid-ask spreads.

1. Introduction

Counterparty risk has played an important role during the 2008 financial crisis. According to the Bank for International Settlements (BIS)[1], two-thirds of counterparty risk losses during the crisis were from counterparty risk adjustments in MtM valuation whereas the rest were due to actual defaults. In order to account for the counterparty risk, recent regulatory changes, such as Basel III, incorporate the counterparty risk adjustments in the calculation of capital requirement. On the other hand, the use of collateral in the derivative market has increased dramatically. According to the survey conducted by the International Swaps and Derivatives Association (ISDA) in 2013[2], the percentage of all trades subject to collateral agreements in the over-the-counter (OTC) market increases from 30 percent in 2003 to 73.7 percent in 2013. OTC market participants continue to adapt collateralization and counterparty risk adjustments in their MtM valuation methodologies for various contracts, including forwards, total return swaps, interest rate swaps and credit default swaps.

When an OTC market participant trades a financial claim with a counterparty, the participant is exposed not only to the price change and default risk of the underlying asset but also to the default risk of the counterparty. To reflect the counterparty default risk in MtM valuation, three adjustments are calculated in addition to the counterparty-risk free value of the claim. While credit valuation adjustment (CVA) accounts for the possibility of the counterparty's default, debt valuation adjustment (DVA) is calculated to adjust for the participant's own default risk. In addition, collateral interest payments and the cost of borrowing generate funding valuation adjustment (FVA). In the industry, the valuation adjustment incorporating CVA, DVA, FVA and collateralization, is called total valuation adjustment (XVA)[3].

In this paper, we study a valuation framework for financial contracts accounting for XVA. We consider two current market conventions for price computation. The main difference in the two conventions rises in the assumption of the liquidation value – either counterparty risk-free value or MtM value with counterparty risk provision – upon default. Brigo, Buescu, and Morini (2012) and Brigo and Morini (2011) show that the values under the two conventions have significant differences and large impacts on net debtors and creditors.

With counterparty risk provision, the MtM value is defined implicitly via a risk-neutral expectation. This gives rise to major challenges in analyzing and computing the contract value. We propose a novel fixed point approach to analyze the MtM value. Our methodology involves solving a sequence of inhomogeneous linear PDEs, whose classical solutions are shown via contraction mapping arguments to converge to the unique fixed point price function. This approach also motivates us to develop an iterative numerical scheme to compute the values of a variety of financial claims under different market conventions.

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[1] See BIS press release at http://www.bis.org/press/p110601.pdf

[2] Survey available at http://www2.isda.org/functional-areas/research/surveys/margin-surveys/

[3] The terminology can be found in Carver (2013) and Capponi (2013), among others.

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In related studies, Fujii and Takahashi (2013) incorporate BCVA and under/over collateralization, and calculate the MtM value by simulation. Henry-Labordère (2012) approximates the MtM value by numerically solving a related nonlinear PDE through simulation of a marked branching diffusion, and provides conditions to avoid a "blow-up" of the simulated solution. Burgard and Kjaer (2011) also consider a similar nonlinear PDE in their study of hedging strategies inclusive of funding costs, and taking into account closeout payments exchanged at the time when either party of the contract defaults. In contrast, our fixed point methodology works directly with the price definition in terms of a recursive expectation, rather than heuristically stating and solving a nonlinear PDE. Our contraction mapping result allows us to solve a series of *linear* PDE problems with bounded classical solutions, and obtain a unique bounded continuous MtM value.

Our model also provides insight on the bid-ask prices of various financial contracts. The XVA is asymmetric for the buyer and the seller. As such, the incorporation of adjustment to unilateral or bilateral counterparty risk leads to a non-zero bid-ask spread. In other words, counterparty risk reveals itself as a market friction, resulting in a transaction cost for OTC trades. In addition, we examine the impact of various parameters such as default rate, recovery rate, collateralization ratio and effective collateral interest rate. We find that a higher counterparty default rate and funding cost reduce the MtM value, whereas the market participant's own default rate and collateralization ratio have positive price effects. For claims with a positive payoff, such as calls and puts, we establish a number of price dominance relationships. In particular, when collateral rates are low, the bid-ask prices are dominated by the counterparty risk-free value. Moreover, the bid-ask prices decrease when we use the MtM value rather than counterparty risk-free value for the liquidation value upon default.

The recent regulatory changes and post-crisis perception of counterparty risk have motivated research on the analysis of XVA. Brigo, Capponi, and Pallavicini (2014) consider an arbitrage-free valuation framework with bilateral counterparty risk and collateral with possible re-hypothecation. Capponi (2013) studies the arbitrage-free valuation of counterparty risk, and analyzes the impact of default correlation, collateral migration frequency and collateral re-hypothecation on the collateralized CVA. Thompson (2010) analyzes the effect of counterparty risk on insurance contracts and examines the moral hazard of the insurers. Brigo and Chourdakis (2009) focus on the valuation of CDS with counterparty risk that is correlated with the reference default. Hull and White (2012) investigate the wrong way risk – the additional risk generated by the correlation between the portfolio return and counterparty default risk.

Let us give an outline of the rest of the paper. In Section 2, we formulate the MtM valuation of a generic financial claim with default risk and counterparty default risks under collateralization. In Section 3, we provide a fixed point theorem and a recursive algorithm for valuation. In Section 4, we compute the MtM values of various defaultable equity claims and derive their bid-ask prices. In Section 5, we apply our model to price a number of defaultable fixed-income claims. Section 6 concludes the paper, and the Appendix contains a number of longer proofs.

2. Model formulation

In the background, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, where $\mathbb{Q}$ is the risk-neutral pricing measure. In our model, there are three defaultable parties: a reference entity, a market participant, and a counterparty dealer. We denote them respectively as parties 0, 1, and 2. The default time τ_i of party $i \in \{0, 1, 2\}$ is modeled by the first jump time of an exogenous doubly stochastic Poisson process. Precisely, we define

$$\tau_i = \inf\left\{t \geq 0 : \int_0^t \lambda_u^{(i)}\, du > E_i\right\}, \tag{2.1}$$

where $\{E_i\}_{i=0,1,2}$ are unit exponential random variables that are independent of the intensity processes $(\lambda_t^{(i)})_{t\geq 0}$, $i \in \{0, 1, 2\}$. Throughout, each intensity process is assumed to be of Markovian form $\lambda_t^{(i)} \equiv \lambda^{(i)}(t, S_t, X_t)$ for some bounded positive function $\lambda^{(i)}(t, s, x)$, and is driven by the *pre-default* stock price S and the stochastic factor X satisfying the SDEs

$$dS_t = (r(t, X_t) + \lambda^{(0)}(t, S_t, X_t))\, S_t\, dt + \sigma(t, S_t)\, S_t\, dW_t, \tag{2.2}$$

$$dX_t = b(t, X_t)\, dt + \eta(t, X_t)\, d\tilde{W}_t. \tag{2.3}$$

Here, $(W_t)_{t\geq 0}$ and $(\tilde{W}_t)_{t\geq 0}$ are standard Brownian motions under $\mathbb{Q}$ with an instantaneous correlation parameter $\rho \in (-1, 1)$. The risk-free interest rate is denoted by $r_t \equiv r(t, X_t)$ for some bounded positive function. At the default time τ_0, the stock price will jump to value zero and remain worthless afterwards. This "jump-to-default model" for S is a variation of those by Merton (1976), Carr and Linetsky (2006), and Mendoza-Arriaga and Linetsky (2011).

2.1. Mark-to-Market value with counterparty risk provision

A defaultable claim is described by the triplet (g, h, l), where $g(S_T, X_T)$ is the payoff at maturity T, $(h(S_t, X_t))_{0\leq t\leq T}$ is the dividend process, and $l(\tau_0, X_{\tau_0})$ is the payoff at the default time τ_0 of the reference entity. We assume continuous collateralization which is a reasonable proxy for the current market where daily or intraday margin calls are common (see Fujii & Takahashi, 2013). For party $i \in \{1, 2\}$, we denote by δ_i the collateral coverage ratio of the claim's MtM value. We use the range $0 \leq \delta_i \leq 120$ percent since dealers usually require over-collateralization up to 120 percent for credit or equity linked notes (see Ramaswamy, 2011, Table 1).

We first consider pricing of a defaultable claim without bilateral counterparty risk. We call this value *counterparty-risk free (CRF) value*. Precisely, the ex-dividend *pre-default* time t CRF value of the defaultable claim with (g, h, l) is given by

$$\Pi(t, s, x) := \mathbb{E}_{t,s,x}\left[e^{-\int_t^T \left(r_u+\lambda_u^{(0)}\right) du} g(S_T, X_T) + \int_t^T e^{-\int_t^u (r_v+\lambda_v^{(0)}) dv} (h(S_u, X_u) + \lambda_u^{(0)} l(u, X_u)) du \right]. \tag{2.4}$$

The shorthand notation $\mathbb{E}_{t,s,x}[\,\cdot\,] := \mathbb{E}[\,\cdot\,|S_t = s,\ X_t = x\,]$ denotes the conditional expectation under $\mathbb{Q}$ given $S_t = s,\ X_t = x$.

Incorporating counterparty risk, we let $\tau = \min\{\tau_0, \tau_1, \tau_2\}$, which is the first default time among the three parties with the intensity function $\lambda(t, s, x) = \sum_{k=0}^2 \lambda^{(k)}(t, s, x)$. The corresponding three default events $\{\tau = \tau_0\}$, $\{\tau = \tau_1\}$ and $\{\tau = \tau_2\}$ are mutually exclusive. When the reference entity defaults ahead of parties 1 and 2, i.e. $\tau = \tau_0$, the contract is terminated and party 1 receives $l(\tau_0, X_{\tau_0})$ from party 2 at time τ_0. When either the market participant or the counterparty defaults first, i.e. $\tau < \tau_0$, the amount that the remaining party gets depends on unwinding mechanism at the default time. We adopt the market convention where the MtM value with counterparty risk provision, denoted by P, is used to compute the value upon the participant's defaults (see Fujii & Takahashi, 2013; Henry-Labordère, 2012).

Throughout, we use the notations $x^+ = x\mathbf{1}_{\{x\geq 0\}}$ and $x^- = -x\mathbf{1}_{\{x<0\}}$. Suppose that party 2 defaults first, i.e. $\tau = \tau_2$. If the MtM value at default is positive $(P_{\tau_2} \geq 0)$, then party 1 incurs a loss only if the contract is under-collateralized by party 2 $(\delta_2 < 1)$ since the amount $\delta_2 P_{\tau_2}^+$ is secured as a collateral. As a result, with the loss rate