European Journal of Operational Research 249 (2016) 647–656

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Stochastics and Statistics

Dynamic mean-risk portfolio selection with multiple risk measures in continuous-time☆

Jianjun Gao[a], Yan Xiong[a], Duan Li[b,*]

[a] Department of Automation, Shanghai Jiao Tong University, China
[b] Department of Systems Engineering & Engineering management, The Chinese University of Hong Kong, Shatin, NT, Hong Kong

ARTICLE INFO

Article history:
Received 14 May 2014
Accepted 3 September 2015
Available online 10 September 2015

Keywords:
Dynamic mean-risk portfolio selection
Conditional value at risk
Safety-first principle
Stochastic optimization
Martingale approach

ABSTRACT

While our society began to recognize the importance to balance the risk performance under different risk measures, the existing literature has confined its research work only under a *static* mean-risk framework. This paper represents the first attempt to incorporate multiple risk measures into *dynamic* portfolio selection. More specifically, we investigate the dynamic mean-variance-CVaR (Conditional value at Risk) formulation and the dynamic mean-variance-SFP (Safety-First Principle) formulation in a continuous-time setting, and derive the analytical solutions for both problems. Combining a downside risk measure with the variance (the second order central moment) in a dynamic mean-risk portfolio selection model helps investors control both a symmetric central risk measure and an asymmetric catastrophic downside risk. We find that the optimal portfolio policy derived from our mean-multiple risk portfolio optimization models exhibits a feature of curved V-shape. Our numerical experiments using real market data clearly demonstrate a dominance relationship of our dynamic mean-multiple risk portfolio policies over the static buy-and-hold portfolio policy.

1. Introduction

One fundamental principle behind the mean-variance (MV) portfolio selection formulation proposed by Markowitz (1952) is to strike a balance between maximizing the expected terminal wealth and minimizing the investment risk. Please refer to Kolm, Tütüncü, and Fabozzi (2014) for a review of the research progress in portfolio optimization during the past 60 years. As a natural generalization of the MV analysis, a general class of mean-risk portfolio optimization models has already become a fundamental instrument in real-world portfolio management. The last half century has witnessed numerous risk measures proposed in the literature. Please see Krokhmal, Zabarankin, and Uryasev (2011) for a comprehensive review. At the same year when Markowitz published his seminal work of the MV formulation, Roy (1952) proposed the fundamental safety-first principle (SFP) in portfolio selection, which concerns the probability that disastrous events happen, a risk measure concentrating on the tail of a distribution, instead of the central moments. Following the same concept of SFP to measure the downside risk, the Value-at-Risk(VaR) is defined as the quantile of the loss under certain confidence level. Although VaR is popular in financial industry (*RiskMetric*^{TM} Morgen, 1996), the VaR has been widely criticized for some of its undesired properties (Tasche, 2002). More specifically, VaR fails to qualify as a coherent risk measure, as it violates some axioms of the coherent risk measures proposed by Artzner, Delbaen, Eber, and Heath (1999). As a modification of the VaR, the conditional Value-at-Risk (CVaR) proposed by Rockafellar and Uryasev (2000) and Rockafellar and Uryasey (2002) is defined as the average value of the loss greater than the VaR for a given confidence level. CVaR has attracted increasing attention in recent years as CVaR is a coherent risk measure proved by Acerbi and Tasche (2002). More importantly, the convexity of the CVaR measure leads to a tractable optimization model of the corresponding mean-CVaR portfolio formulation and its extensions, see, e.g., Alexander and Baptista (2002), Rockafellar and Uryasey (2002), Zhu and Fukushima (2009) and Huang, Zhu, Fabozzi, and Fukushima (2010). While almost all mean-risk frameworks in the literature adhere to a single risk measure, Roman, Darby-Dowman, and Mitra (2007) proposed recently a static portfolio selection model with a combined risk measure of both variance and CVaR, which offers a new way of thinking under the mean-risk framework.

☆ This research work was partially supported by Research Grants Council of Hong Kong under grant 414513, by National Natural Science Foundation of China under grant 71201102 and 61573244 by the Ph.D. Programs Foundation of Ministry of Education of China under grant 20120073120037. The third author is also grateful to the support from Patrick Huen Wing Ming Chair Professorship of Systems Engineering & Engineering Management.

* Corresponding author. Tel.: +(852) 3943- 8323; fax: +(852) 2603- 5505.
E-mail address: dli@se.cuhk.edu.hk (D. Li).

http://dx.doi.org/10.1016/j.ejor.2015.09.005

Various dynamic mean-risk portfolio optimization models have been developed according to different risk measures. The past 15 years has witnessed rapid progresses of the dynamic MV formulation by leaps and bounds, see Li and Ng (2000), Zhou and Li (2000), Li, Zhou, and Lim (2001), Zhu, Li, and Wang (2004), Bielecki, Jin, Pliska, and Zhou (2005), Basak and Chabakauri (2010), Cui, Li, Wang, and Zhu (2012), Cui, Gao, Li, Li, (2014). The earliest work on dynamic mean-SFP portfolio selection model by Li, Chan, and Ng (1998) approximates the SFP measure by using the Bienaymé–Tchebycheff Inequality as did in the original work of Roy (1952), which in turn leads back to a dynamic MV formulation. However, when the underlying distribution of the random return is not normally distributed and is asymmetric, such an approximation method is certainly not accurate enough and basically loses the essential feature of the safety-first principle. As for the mean-CVaR portfolio models, the current state-of-the-art still remains mainly at the level of static formulations which generate optimal buy-and-hold portfolio policies, e.g., see Rockafellar and Uryasev (2000), Alexander and Baptista (2002), Rockafellar and Uryasey (2002), Zhu and Fukushima (2009), Künzi-Bay and János (2006) and Huang et al. (2010). Different from MV portfolio optimization where only the first two central moments of the distribution are needed, the distribution information of the returns is always required in computing the CVaR of the portfolio loss. Thus, either an explicit distribution function or a discrete distribution is assumed in mean-CVaR portfolio optimization. The past several years have witnessed efforts to extend static mean-CVaR optimization models to two-period or multi-period settings, see, e.g., Fábián (2006), Fábián and Szoke (2007), Hibiki (2006). However, due to the assumption of the discrete scenario of the returns, these models often lead to a computational intractability while the number of scenario and/or periods is only relatively large. By using the dynamic programming approach, Siegmann and Lucas (2005) studied the loss aversion based dynamic portfolio allocation model which is related to mean-downside risk portfolio optimization. As for continuous-time models, Jin, Yan, and Zhou (2005) showed that a general class of mean-downside risk portfolio optimization problem in continuous-time setting is ill-posed in the sense that the optimal value cannot be achieved. Despite of such a negative result, there do exist research works that integrate the downside risk measure into the utility maximization model in continuous-time, e.g., Basak and Shapiro (2001) considered the VaR constraint on the terminal wealth, and Yiu (2004) extended such a result to the case when the VaR constraint is imposed for the entire investment process. Chiu, Wong, and Li (2012) modified the original mean-SFP formulation reasonably by imposing an upper bound on the terminal wealth level and completely solved the continuous-time mean-SFP asset-liability problem.

Different risk measures emphasize different aspects of the random investment loss. While the variance measures the deviation of the random variable from the expected value, Roy's safety-first principle (SFP) (Roy, 1952) focuses on the extreme events in the tail part of the random return distribution. While SFP is a probability measure, the CVaR is a conditional expectation conditioned on that losses are greater than the VaR for a given confidential level. If we examine the investment performance according to different spectra of the risk measures, any policy generated from a mean-risk portfolio model with a sole risk measure may not become a good choice. Roman et al. (2007) showed that the optimal policy generated from the mean-CVaR portfolio model could induce a very large variance, which leads to a small Sharpe ratio. The confidential level of the loss in CVaR is usually set at 95 or 99 percent. Intuitively speaking, the CVaR focuses on the 5 or 1 percent tail part of the random return distribution, which neglects the risk exhibited in other parts of the distribution. On the other hand, the CVaR of the portfolio generating from the traditional MV model could be also unacceptably large. To overcome the inconsistence between the policies generated from the MV and mean-downside risk models, Roman et al. (2007) proposed to combine the CVaR and the variance together as two risk measures in a multi-objective portfolio optimization problem. After solving these static optimization problems, Roman et al. (2007) showed the advantage of these portfolio models with multiple risk measures by checking them against various market data sets.

Motivated by the work in Roman et al. (2007) under a static setting, we consider in this paper its extension to dynamic portfolio selection with multiple risk measures. More specifically, we consider in this work two kinds of dynamic mean-multiple risk portfolio optimization models, namely, the dynamic mean-variance-CVaR (MVC) and the dynamic mean-variance-SFP (MVS) portfolio optimization problems. To our knowledge, this is the first work in the literature studying dynamic mean-multiple risk portfolio optimization models. Compared to the existing literature, our contributions include several significant features. Under our market setting, we are able to derive the analytical forms of the portfolio policy for both MVC and MVS portfolio optimization models. Due to the combined risks in the objective function, our multiple-risks based portfolio model conquers the ill-posedness of the mean-downside risk portfolio optimization model with a sole downside risk (Jin et al., 2005). We also reveal a key difference of our portfolio model with the well known dynamic MV portfolio policy, e.g., see Li and Ng (2000), Zhou and Li (2000), Bielecki et al. (2005). The optimal portfolio policies derived from both MVC and MVS exhibit a curved V-shape property, i.e., with respect to a certain level of the current wealth, both of the MVC and MVS portfolio policies increase the allocation to the risky assets in both directions. Compared with the static MVC model proposed by Roman et al. (2007), our newly derived dynamic MVC portfolio policy can reduce both the variance and the CVaR measures significantly.

The remaining of this paper is organized as follows. We present the market setting and problem formulations in Section 2. We investigate and solve completely the dynamic MVC and dynamic MVS formulations in Section 3 and Section 4, respectively. In Section 5, we compare the dynamic MVC and MVS portfolio policies with the dynamic MV portfolio policy via two illustrative examples. We also compare our dynamic MVC portfolio model with the static buy-and-hold portfolio policy. Throughout of the paper, we use notation $\mathbf{1}_{\mathcal{A}}$ for the indicator function, i.e., $\mathbf{1}_{\mathcal{A}} = 1$ if condition $\mathcal{A}$ holds and $\mathbf{1}_{\mathcal{A}} = 0$ otherwise, Q' for the transpose of matrix Q, and $(y)^+$ for the nonnegative part of y, i.e., $(y)^+ \triangleq \max\{0, y\}$. We denote normal random variable X with mean a and variance b by $X \sim \mathcal{N}(a, b)$, and denote the probability density function and the cumulative distribution function (CDF) of the standard normal variable by $\phi(\cdot)$ and $\Phi(\cdot)$, respectively. For any particular optimization problem $(\mathcal{P})$, we use $v(\mathcal{P})$ to denote its optimal objective value.

2. Market setting and problem formulation

We consider a market with n risk assets and one risk free asset which can be traded continuously within a time horizon $[0, T]$. An investor enters the market with initial wealth x_0 and allocates his wealth in these $n+1$ assets continuously. The price process $S_0(t)$ of the risk free asset follows the ordinary differential equation, $dS_0(t) = r(t)S_0(t)dt,\ t \in [0, T]$, with $S_0(0) = s_0$, where $r(t)$ is the risk free return, $0 \le t \le T$. In this work, all the randomness is modeled by a complete filtrated probability space $\{\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0}\}$, on which an $\mathcal{F}_t$ adapted n-dimensional Brownian motion $W(t) = \left(W^1(t), \ldots, W^n(t)\right)'$ is defined, where $W^i(t)$ and $W^j(t)$ are mutually independent for $i \neq j$. The price processes of the n risky assets satisfy the following set of stochastic differential equations (SDE),

$$dS_i(t) = S_i(t)\left(\mu_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dW^j(t)\right), \quad i = 1, \ldots, n,$$

with $S_i(0) = s_i$, where $\mu_i(\cdot)$ and $\sigma_{ij}(\cdot)$ are the appreciation and volatility, respectively. We assume that $r(t)$, $\mu_i(\cdot)$ and $\sigma_{ij}(\cdot)$ are $\mathcal{F}_t$-