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Decision Support

Accurate algorithms for identifying the median ranking when dealing with weak and partial rankings under the Kemeny axiomatic approach

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ABSTRACT

Preference rankings virtually appear in all fields of science (political sciences, behavioral sciences, machine learning, decision making and so on). The well-known social choice problem consists in trying to find a reasonable procedure to use the aggregate preferences or rankings expressed by subjects to reach a collective decision. This turns out to be equivalent to estimate the consensus (central) ranking from data and it is known to be a NP-hard problem. A useful solution has been proposed by Emond and Mason in 2002 through the Branch-and-Bound algorithm (BB) within the Kemeny and Snell axiomatic framework. As a matter of fact, BB is a time demanding procedure when the complexity of the problem becomes untractable, i.e. a large number of objects, with weak and partial rankings, in presence of a low degree of consensus. As an alternative, we propose an accurate heuristic algorithm called FAST that finds at least one of the consensus ranking solutions found by BB saving a lot of computational time. In addition, we show that the building block of FAST is an algorithm called QUICK that finds already one of the BB solutions so that it can be fruitfully considered to speed up even more the overall searching procedure if the number of objects is low. Simulation studies and applications on real data allows to show the accuracy and the computational efficiency of our proposal.

1. Introduction

The consensus ranking problem, also known as social choice problem, arises any time n subjects (or *judges*) are asked to express their preferences on a set of m objects. These objects are placed in order by each subject (where 1 represents the best and m the worst) without any attempt to describe how much one differs from the others or whether any of the alternatives is good or acceptable. Every independent observation is a permutation of m distinct positive integer numbers. To be more specific, when the subject assigns the integer values from 1 to m to all the m items we have a *complete* (or full) ranking. Whenever instead the judge fails to distinguish between two or more items and assigns to them the same integer number (expressing indifference to the relative order of this set of items), we deal with *tied* (or weak) rankings. Moreover we have a *partial* ranking when judges are asked to rank a subset of the entire set of objects (e.g. pick the three most favorite items out of a set of five) (D'Ambrosio & Heiser, 2014; Marden, 1996). Rankings are by nature peculiar data in the sense that the sample space of m objects can be only visualized in a $(m-1)$-dimensional hyperplane by a discrete structure that is called the *permutation polytope*, S_m. A polytope is a convex hull of a finite set of points in $\mathbb{R}^m$ (Heiser, 2004; Thompson, 1993). For example the space considering 4 objects with all possible ties is a truncated octahedron that can be visualized in Fig. 1 (Heiser & D'Ambrosio, 2013). As we already pointed out, the permutation polytope is inscribed in a $(m-1)$-dimensional subspace, hence, for $m > 4$, such structures are impossible to visualize. The permutation polytope is the natural space for ranking data. To define it no data are required, it is completely determined by the number of items involved in the preference choice; data add only information on which rankings occur and with what frequency they occur. This space is discrete and finite. It is characterized by symmetries and it is endowed with a graphical metric.

The problem of combining rankings to obtain a ranking representative of the group has been studied by numerous researchers in several areas, e.g. voting systems, economics, machine learning, psychology, political sciences, for more than two centuries. In the framework of distance-based models for rankings, searching for consensus ranking is a very important step in modeling the ranking process (Marden, 1996). These models are usually exponential family models (Diaconis, 1988) and they are completely specified by two parameters, a dispersion parameter and a consensus (central) ranking.

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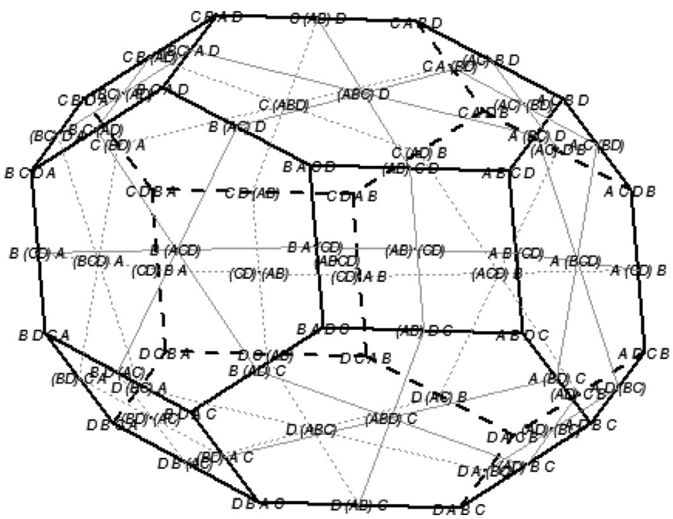

Fig. 1. Permutation polytope for all full and weak ranking for four objects. For every ranking the correspondent ordering is shown.

Maximum likelihood estimates of the dispersion parameter assume the knowledge of the central ranking. When the consensus ranking is not known it should be estimated. Unfortunately, even if there are close formulas for this estimation they are not feasible because of the complexity of the problem (Critchlow, Fligner, & Verducci, 1991; Critchlow, 1985; Diaconis, 1988; Fligner & Verducci, 1986, 1988). Several methods to aggregate individual preference rankings have been proposed since the works of Arrow (1951) Barthelemy and Monjardet (1981) Black (1958) Bogart (1973) Cook and Seiford (1978) Coombs (1964) Davis, DeGroot, and Hinich (1972) de Borda (1781) de Condorcet (1785) Emond and Mason (2002) Goodman and Markowitz (1952) and Meila, Phadnis, Patterson, and Bilmes (2012).

In this paper, we propose two heuristic algorithms called QUICK and FAST to derive the consensus ranking from the aggregation of individual preferences within the Kemeny and Snell axiomatic framework. Both algorithms can be viewed as alternatives to the branch-and-bound algorithm by Emond and Mason. The BB algorithm turns out to be a time consuming procedure when the number of objects is high and especially when the internal degree of consensus present in the data is weak. Both QUICK and FAST algorithms can deal with complete and tied rankings as well as with incomplete (or partial) rankings. As a matter of fact, the QUICK algorithm is the building block of the FAST algorithm. Both provide savings in computational time, but the FAST algorithm is more accurate because it finds more than one of the solutions found by the BB algorithm and it can also easily deal with problems characterized by a large number of objects to be ranked and weak and partial rankings and/or a low degree of internal consensus. On the other hand, the QUICK algorithm turns out to be really useful when the number of objects is limited because it returns one of the solutions found by the BB, or a really close solution, in a considerably small amount of time.

The rest of the paper is organized as follow. In Section 2 we briefly present some of the proposed approaches to aggregate preference rankings and derive a consensus. In Section 3 we describe the branch-and-bound algorithm by Emond and Mason. Section 4 is devoted to describe the proposed algorithms, then in Sections 5 and 6 we present a simulation study and applications on real data to evaluate both the accuracy and the efficiency of our proposal. Concluding remarks are then found in Ssection 7.

2. Finding the consensus ranking, some approaches

The term consensus ranking is a generic name for any ranking that summarizes a set of individual rankings. There exist two broad classes

Table 1
Example data to illustrate Borda's method of marks.

# Voters	Alternatives		
	A	B	C
12	2	1	3
5	1	2	3
7	3	2	1

Table 2
Support table to illustrate Condorcet's method on example data

	A	B	C
A	–	5	17
B	19	–	17
C	7	7	–

of approaches to aggregate preference rankings in order to find a consensus (Cook, 2006):

- *ad hoc methods*, which can be divided into elimination (for example the American system method, the pairwise majority rule, etc.) and non-elimination (for example Borda's methods of marks (1781), Condorcet's method (1785), etc.);
- *distance-based models*, according to which it is necessary to define a distance of the desired consensus from the individual rankings.

A more detailed description of both these approaches can be found in Cook (2006).

How to aggregate subjects preferences to create a consensus is a problem that goes back to 1781 when Borda formulated the method of marks (also known as *Borda's count*) for determining the winner in elections with more than 2 candidates. This method is quite simple and it is based on calculating the total rank for each alternative. For example, if we consider the rankings in Table 1 the total rank for each alternative is given by:

- $A = 12 \times 2 + 5 \times 1 + 7 \times 3 = 50$,
- $B = 12 \times 1 + 5 \times 2 + 7 \times 2 = 36$,
- $C = 12 \times 3 + 5 \times 3 + 7 \times 1 = 58$,

resulting in the consensus (*BAC*). Borda's method of marks was criticized by Condorcet, which proposed to use the majority rule on all the pairwise comparisons between alternatives. Condorcet's solution for the rankings reported in Table 1 can be obtained by calculating the support obtained by every pairwise comparison between options, reported in Table 2. From Table 2 we can deduce that $B \succ A$, $B \succ C$ and $A \succ C$, resulting also in the consensus ranking (*BAC*). In applying this method, unfortunately, one problem can be encountered, i.e. if intransitive preferences occur the simple majority procedure breaks down (*paradox of voting* (Arrow, 1951), according to which a set of transitive preferences can generate a global intransitive preference as group preference).

In the last century the rank aggregation problem has been approached from a statistical perspective. Kendall (1938) was the first to propose a method to aggregate input rankings to find a consensus. He studied the consensus problem as a problem of estimation and he proposed to rank items according to the mean of the ranks assigned, thus proposing a method equivalent to Borda's one. Moreover he suggested to consider the Spearman rank correlation coefficient ρ, that, given two preference rankings R and R^*, is defined as:

$$\rho = 1 - \frac{6 \sum_{i=1}^{n} d_i^2}{n^3 - n}, \qquad (1)$$

where $d_i^2(R, R^*) = \sum_{j=1}^{m} (R_j - R_j^*)^2$ is the squared difference between rankings R and R^* (Kendall, 1948, page 8). The Spearman's ρ is equivalent to the product moment correlation coefficient and it treats rankings as they are scores summing the square of ranked differences.