European Journal of Operational Research 249 (2016) 751–770

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Interfaces with Other Disciplines

A dynamic program for valuing corporate securities

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ARTICLE INFO

Article history:
Received 9 January 2015
Accepted 15 October 2015
Available online 28 October 2015

Keywords:
Option theory
Structural models
Corporate securities
Corporate bonds
Dynamic programming

ABSTRACT

We design and implement a dynamic program for valuing corporate securities, seen as derivatives on a firm's assets, and computing the term structure of yield spreads and default probabilities. Our setting is flexible for it accommodates an extended balance-sheet equality, arbitrary corporate debts, multiple seniority classes, and a reorganization process. This flexibility comes at the expense of a minor loss of efficiency. The analytical approach proposed in the literature is exchanged here for a quasi-analytical approach based on dynamic programming coupled with finite elements. To assess our construction, which shows flexibility and efficiency, we carry out a numerical investigation along with a complete sensitivity analysis.

1. Introduction

The option-based approach for valuing corporate bonds goes back to Merton (1974). He considers a model for a firm with a simple capital structure made of a pure bond and a common stock (equity). He interprets the stock as a European call option on the firm's assets, whose value follows a geometric Brownian motion, as set by Black and Scholes (1973). The option's expiry date and strike price are the bond's maturity date and principal amount, respectively. Then, he evaluates the debt and equity in closed form. Merton's pioneering paper has given rise to an extensive literature, known as the *structural model*, where corporate securities are expressed as derivatives on the firm's asset value. The default event at a given payment date occurs when the state variable falls under a certain *default barrier*.

Black and Cox (1976) extend Merton's model in two ways, and solve the structural model in closed form. They propose an exogenous default barrier to cover the pure bond's holders against severe decreases on the firm's asset value before maturity. They also consider uncovered bond portfolios made of a pure senior bond and a pure junior bond with the same maturity. Geske (1977) uses the theory of compound options, and further extends Merton's model to arbitrary corporate-bond portfolios. However, his analytical approach remains questionable when the number of coupon dates increases. Leland (1994) considers a continuous coupon perpetuity, which results in a constant default barrier over time. Next, by maximizing the present value of equity, he solves for the so-called *endogenous default barrier* in closed form. He accounts for tax benefits under the survival event and bankruptcy costs under the default event. These frictions allow Leland to discuss the notions of maximum debt capacity and optimal capital structure. The latter is a break-down of the Modigliani–Miller conjecture, which states that, in pure and perfect capital markets, the firm's asset value is independent of its capital structure.

The aim of this paper is to design and implement a unified dynamic-programming framework for valuing corporate securities and computing the term structure of yield spreads and default probabilities. This model extends Merton (1974), Black and Cox (1976), Geske (1977), and Leland (1994), for it accommodates arbitrary corporate debts, multiple seniority classes, payouts, tax benefits, bankruptcy costs, and a reorganization process. The latter is fulfilled through an augmentation of the state process, which now includes not only the firm's asset value but also the number of grace periods called for by the firm before the current date. The reorganization and liquidation (default) barriers inferred at payment dates are completely endogenous, and follow from an optimal decision process. These extensions come at the expense of a minor loss of efficiency. The analytical approach of the above-mentioned authors is exchanged here for a quasi-analytical approach based on dynamic programming coupled with finite elements.

Further extensions to Merton's seminal paper in the literature are twofold. The first research stream assumes a very simple firm's capital structure and solves the structural model in closed form. Longstaff and Schwartz (1995), then Briys and de Varenne (1997), consider a pure bond in Black and Cox' setting with a Gaussian risk-

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http://dx.doi.org/10.1016/j.ejor.2015.10.026
0377-2217/

Table 1a
Value functions at $t_n \in \mathcal{P}$ without a reorganization process.

Balance-sheet components	Default: $a \le b_n^*$		Survival : $a > b_n^*$
	$(0, b_n^{**}]$	$[b_n^{**}, b_n^*]$	
$a = A_{t_n}$	a	a	a
$\text{TB}_{t_n}(a)$	0	0	$\text{TB}_{t_n^+}(a') + \text{tb}_n$
$-\text{BC}_{t_n}(a)$	$-wa$	$-wa$	$-\text{BC}_{t_n^+}(a')$
$=$	$=$	$=$	$=$
$D_{t_n}^s(a)$	$a(1-w)$	$D_{t_n^+}^s(a) + d_n^s$	$D_{t_n^+}^s(a') + d_n^s$
$D_{t_n}^j(a)$	0	$a(1-w) - D_{t_n}^s(a)$	$D_{t_n^+}^j(a') + d_n^j$
$S_{t_n}(a)$	0	0	$S_{t_n^+}(a') - d_n$

Table 1b
Value functions at $t_n \in \mathcal{P}$ with a reorganization process.

Balance-sheet components	Liquidation: $a \le b_n^l$		Reorganization	Survival
	$(0, b_n^{**}]$ and $g \le \bar{g}$	$\left[b_n^{**}, b_n^l\right]$ and $g \le \bar{g}$	$b_n^l \le a \le b_n^r$ and $g < \bar{g}$	$a > b_n^r$ and $g \le \bar{g}$
$a = A_{t_n}$	a	a	a	a
$\text{TB}_{t_n}(a, g)$	0	0	$\text{TB}_{t_n^+}(a'', g+1) +$ $(1-\eta)\text{tb}_n$	$\text{TB}_{t_n^+}(a', g) +$ tb_n
$-\text{BC}_{t_n}(a, g)$	$-wa$	$-wa$	$-\text{BC}_{t_n^+}(a'', g+1)$	$-\text{BC}_{t_n^+}(a', g)$
$-\text{RC}_{t_n}(a, g)$	0	0	$-w_r a -$ $\text{RC}_{t_n^+}(a'', g+1)$	$-\text{RC}_{t_n^+}(a', g)$
$=$	$=$	$=$	$=$	$=$
$D_{t_n}^s(a)$	$a(1-w)$	$D_{t_n^+}^s(a, g) +$ d_n^s	$D_{t_n^+}^s(a'', g+1) +$ $(1-\eta)d_n^s$	$D_{t_n^+}^s(a', g) +$ d_n^s
$D_{t_n}^j(a)$	0	$a(1-w) -$ $D_{t_n}^s(a, g)$	$D_{t_n^+}^j(a'', g+1) +$ $(1-\eta)d_n^j$	$D_{t_n^+}^j(a', g) +$ d_n^j
$S_{t_n}(a)$	0	0	$S_{t_n^+}(a'', g+1) -$ $(1-\eta)d_n$	$S_{t_n^+}(a', g) -$ d_n

free rate. Their papers differ on the form of their exogenous default barriers. Collin-Dufresne and Goldstein (2001) extend Longstaff and Schwartz's setting and allow for a Gaussian exogenous default barrier. Along the same lines, Hsu, Saá-Requejo, and Santa-Clara (2010) evaluate the pure bond in closed form when the firm's asset value is completely independent from the risk-free rate, which moves as a square-root process. In case of dependence, they use Monte Carlo simulation for valuation purposes. Nivorozhkin (2005a; 2005b) introduce bankruptcy costs for uncovered bond portfolios in Black and Cox' setting. Starting from Leland's framework, Leland and Toft (1996) consider a finite-maturity coupon bond that is renewed as long as the firm's survives. Chen and Kou (2009) exchange the pure-diffusion dynamics of the firm's asset value of Leland and Toft for a double-exponential jump-diffusion process. Then, they evaluate the debt in closed form. Several authors extend the analytical approach to analyze reorganization processes, especially under Black and Cox-like settings (Abinzano, Seco, Escobar, & Olivares, 2009; Ericsson & Reneby, 1998) or Leland-like settings (Bruche & Naqvi, 2010; François & Morellec, 2004; Mella-Barral & Perraudin, 1997; Shibata & Tian, 2012).

The second research stream refers to numerical methods: numerical integration (Moraux, 2004), finite differences (Anderson, Sundaresan, & Tychon, 1996; Brennan & Schwartz, 1978; Fan & Sundaresan, 2000), binomial trees (Anderson & Sundaresan, 1996; Broadie, Chernov, & Sundaresan, 2007; Broadie & Kaya, 2007), dynamic programming (Annabi, Breton, & François, 2012a; 2012b), and Monte Carlo simulation (Galai, Raviv, & Wiener, 2007; Zhou, 2001). Except for Brennan and Schwartz (1978) and Zhou (2001), the rest of these papers consider reorganization processes. Our paper differs from this body of literature in that it handles an arbitrary debt portfolio and several endogenous barriers for monitoring the reorganization process, depending on the firm's capital structure, debt seniority, and reorganization design. Most of these papers build on Black and Cox (1976) or Leland (1994) assumptions. Although Broadie et al. (2007) assume a cash-flow-based framework and a consol debt, their binomial tree resembles to our dynamic program in that it handles a double endogenous barrier. Finally, our model differs from Annabi et al. (2012a; 2012b) in two ways. Firstly, as claimed earlier, our setting accommodates arbitrary debt portfolios, while theirs assumes a continuous perpetuity. Secondly, our reorganization process focuses on the optimal number of grace periods a firm can call for, while theirs focuses on the negotiation in between claimholders under default.

Among other objectives, the structural model attempts to explain the observed yield spreads and default frequencies. Despite its parsimony, the simplest structural model (Merton, 1974) compares extremely well to the classic statistical approach for bankruptcy prediction (Hillegeist, Keating, Cram, & Lundstedt, 2004), and, to a lesser extent, to the neural-network approach (Aziz & Dar, 2006). A hybrid approach can be developed, where some of the statistical risk factors are inferred from the structural model, e.g. the distance to default (Benos & Papanastasopoulos, 2007). More complex structural models have further explained the observed yield spreads and default frequencies (Collin-Dufresne & Goldstein, 2001; Delianedis & Geske, 2001; 2003; Eom, Helwege, & Huang, 2004; Huang & Huang, 2012; Leland, 2004; Suo & Wang, 2006). According to Delianedis and Geske (2001), the most important components of credit risk are default, recovery, tax benefits, jumps, liquidity, and market factors.

On the one hand, closed-form solutions, where available, are obviously preferred to approximations. They are extremely efficient; they assure the highest accuracy at a very low computing time. Closed-form solutions explicitly link the unknown parameters to their input parameters and, thus, allow for a direct sensitivity analysis. However, they rely on very simplified assumptions. On the other hand, more realistic models are solved by means of numerical procedures. Our dynamic program is an acceptable compromise in terms of flexibility and efficiency.

Dynamic programming is widely used for modeling and solving several optimal Markov decision processes in finance. Examples include Kraft and Steffensen (2013) and Fu, Wei, and Yang (2014) for optimal portfolios and Gamba and Triantis (2008) for financial flexibilities.

This paper is organized as follows. Section 2 presents the model and provides several properties of the debt- and equity-value functions, and Section 3 proposes a reorganization process. Section 4 is a numerical investigation, which replicates reported results from the literature and carries out a complete sensitivity analysis. Section 5 concludes. The dynamic program is resolved in the Appendix A.

2. Model and notation

Consider a public company with the following capital structure: a portfolio of senior and junior bonds and a residual claim, that is, a common stock (equity). Let $\mathcal{P} = \{t_0, t_1, \ldots, t_n, \ldots, t_N = T\}$ be a set of payment dates. At time $t_n \in \mathcal{P}$, the firm is committed to paying $d_0^s + d_0^j = d_0 \ge 0$ and $d_n^s + d_n^j = d_n > 0$, for $n = 1, \ldots, N$, to its creditors (bondholders), where d_n^s and d_n^j are the outflows generated at t_n by the senior and junior bonds, respectively. The total outflow d_n includes interest as well as principal payments. The interest payments are indicated by d_n^{int}. The amounts d_n^s, d_n^j, and d_n^{int}, for $n = 0, \ldots, N$, are known to all investors from the very beginning. The last payment dates of the senior and junior debts, both in $\mathcal{P}$, are indicated by T^s and T^j, respectively. Several authors consider a senior coupon bond and a junior coupon bond with a longer maturity, that is, $0 \le T^s < T^j = T$. Senior bondholders are therefore assured payment before junior bondholders. This realistic case is embedded in our setting.

Assume that the firm's asset value $\{A\}$ is lognormal with $A_0 > 0$, for $t \in [0, T]$. The present value of tax benefits, bankruptcy costs, senior