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Continuous Optimization

Multiparametric linear programming: Support set and optimal partition invariancy

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ABSTRACT

Traditional sensitivity and parametric analysis in linear optimization was based on preserving optimal basis. Interior point methods, however, do not converge to a basic solution (vertex) in general. Recently, there appeared new techniques in sensitivity analysis, which consist in preserving so called support set invariancy and optimal partition invariancy. This paper reflects the renascence of sensitivity and parametric analysis and extends single-parametric results to the case when there are multiple parameters in the objective function and in the right-hand side of equations. Multiparametric approach enables us to study more complex perturbation occurring in linear programs than the simpler sensitivity analysis does. We present a description of the set of admissible parameters under the mentioned invariances, and compare them with the classical optimal basis concept.

1. Introduction

Sensitivity analysis and parametric programming are basic tools for studying perturbations in optimization problems and they are still in focus of research even for linear programming problems; the recent papers includes e.g. Borrelli et al. (2003), Greenberg (2000), Hadigheh et al. (2007, 2008), Hadigheh and Terlaky (2006a,b), Stallaert (2007), and Wendell (2004). Perturbations occur due to measurement errors or just to answer managerial questions "What if…". Naturally, more complex perturbations need to be modelled by a number of parameters. In the past, only few authors were concerned with multiparametric programming – Borrelli et al. (2003), Filippi (2004), Gal (1979), Ward and Wendell (1990) among others. So far, various kinds of invariances (Dehghan et al., 2007; Hadigheh and Terlaky, 2006a,b; Greenberg, 1994) were used mainly in single-parameter or bi-parametric analysis. Our aim is to extend them to the multiparametric case.

Let us consider the linear program

$$\min\ c^Tx \quad \text{subject to } Ax = b,\ x \geqslant 0 \tag{P}$$

and its dual in the form

$$\max\ b^Ty \quad \text{subject to } A^Ty \leqslant c, \tag{D}$$

where $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. By

$$\mathscr{P} := \{x | Ax = b,\ x \geqslant 0\},$$
$$\mathscr{D} := \{y | A^Ty \leqslant c\},$$

we denote the feasible sets of the primal and dual problems, respectively, and by $\mathscr{P}^*$ and $\mathscr{D}^*$ the corresponding optimal solution sets. The support set of a nonnegative vector x is defined as

$$\sigma(x) := \{i | x_i > 0\}.$$

The index set $\{1,\dots,n\}$ can be disjointly partitioned into two subsets

$$\mathscr{B} := \{i | x_i > 0 \text{ for some } x \in \mathscr{P}^*\},$$
$$\mathscr{N} := \{i | c_i - A_{\cdot i}^Ty > 0 \text{ for some } y \in \mathscr{D}^*\},$$

which is known as *optimal partition* (Dehghan et al., 2007; Greenberg, 2000; Hadigheh et al., 2007, 2008; Hadigheh and Terlaky, 2006a,b; Greenberg, 1994; Jansen et al., 1997) and is unique. A primal feasible vector x^0 and a dual feasible vector y^0 are called *a pair of strictly complementary solutions* if they satisfy $(A^Ty^0 - c)^Tx^0 = 0$ and $x^0 + c - A^Ty^0 > 0$. Such a pair always exists provided that both the primal and dual problems are feasible; see Hadigheh et al. (2007, 2008), Hadigheh and Terlaky (2006b), Greenberg (1994, 1997) and references there. Clearly, $\sigma(x^0) = \mathscr{B}$.

Let μ be a k-dimensional vector of parameters. We introduce the general parametrization of the primal problem (P) by

$$\min\ c(\mu)^Tx \quad \text{subject to } A(\mu)x = b(\mu),\ x \geqslant 0, \tag{P$_\mu$}$$

that is, the matrix $A(\mu)$ and vectors $b(\mu), c(\mu)$ depend on the vector of parameters μ. Its dual is

$$\min\ b(\mu)^Ty \quad \text{subject to } A(\mu)^Ty \leqslant c(\mu). \tag{D$_\mu$b)}$$

Let μ^0 be fixed and let x^* and y^* be optimal solutions of (P_{μ^0}) and (D_{μ^0}), respectively. The optimal partition corresponding to μ^0 is denoted by $(\mathscr{B}, \mathscr{N})$.

We follow Dehghan et al. (2007), Hadigheh and Terlaky (2006b) and categorize parametric analysis in three types of invariances:

- *Optimal basis invariancy:* Suppose that x^* is a basic optimal solution associated with the basis B. We want to compute the set $\Upsilon_B(x^*)$ of parameters μ for which B remains the optimal basis.

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Nomenclature

$A_{i\cdot}$	the ith row of a matrix A	$\mathrm{cone}\,\mathscr{M}$	convex cone generated by a set $\mathscr{M}$
$A_{\cdot j}$	the jth column of a matrix A	$\mathrm{cl}\,\mathscr{M}$	closure of a set $\mathscr{M}$
$u \lneqq v$	at once $u \leqslant v$ and $u \neq v$	$\mathrm{relint}\,\mathscr{M}$	relative interior of a set $\mathscr{M}$
A_P	submatrix of A consisting of the columns indexed by P	$\mathscr{L}^{\perp}$	orthogonal complement of a linear space $\mathscr{L}$
A_P^T	transposition of A_P, i.e. $(A_P)^T$		

- *Support set invariancy:* The aim is to compute the set of parameters μ such that there is an optimal solution x_μ^* of (P_μ) for which $\sigma\left(x_\mu^*\right) = \sigma(x^*)$. Moreover, more general cases can be considered:

  1. $\sigma\left(x_\mu^*\right) = \sigma(x^*)$,
  2. $\sigma\left(x_\mu^*\right) \supseteq \sigma(x^*)$,
  3. $\sigma\left(x_\mu^*\right) \subseteq \sigma(x^*)$.

  The corresponding sets of parameters are denoted by $\Upsilon_1(x^*)$, $\Upsilon_2(x^*)$ and $\Upsilon_3(x^*)$, respectively. We refer to Hadigheh and Terlaky (2006b) for the economic interpretation of these cases.
- *Optimal partition invariancy: We* want to compute the set Υ_p of parameters μ for which the problem (P_μ) has the optimal partition $(\mathscr{B},\mathscr{N})$.

The sets $\Upsilon_B(x^*), \Upsilon_1(x^*), \Upsilon_2(x^*), \Upsilon_3(x^*)$, and Υ_p are referred to as *critical regions*, corresponding to particular invariances.

Optimal basis invariancy (Gal, 1979; Gal and Greenberg, 1997; Ward and Wendell, 1990; Gál and Nedoma, 1971) is used when solving linear programs by the simplex method. The other kinds of invariances appeared with the birth of interior point methods solving linear programs in polynomial time. These methods yield nonbasic feasible solutions in general, and so new kinds of invariances had to be considered. There was studied optimal partition invariancy (Borrelli et al., 2003; Greenberg, 2000; Hadigheh et al., 2008; Hadigheh and Terlaky, 2006b; Greenberg, 1994; Berkelaar et al., 1997; Jansen et al., 1997) and recently appeared support set invariancy (Dehghan et al., 2007; Hadigheh et al., 2007; Hadigheh and Terlaky, 2006a,b). Both of them also overcome the main drawback of optimal basis invariancy – possible degeneracy of the optimal solution.

We focus on two leading types of parametrization: vector of parameters in the objective function and in the right-hand side of equations. For both types we propose a description of all mentioned types of critical regions, and show relationships between them.

2. Objective function perturbation

Let us consider the special case of parametrization, when parameters are situated in the objective function, i.e.

$$\min\ \lambda^T x \quad \text{subject to } Ax = b,\ \ x \geqslant 0, \tag{P_λ}$$

where λ is the n-dimensional vector of parameters. The dual problem is

$$\max\ b^T y \quad \text{subject to } A^T y \leqslant \lambda. \tag{D_λ}$$

Let $\lambda^0 \in \mathbb{R}^n$ be fixed and x^* be an optimal solution of (P_{λ^0}). In the following, we give a description of critical regions for all proposed kinds of invariances with the starting point λ^0.

2.1. Optimal basis invariancy

Suppose that x^* is an optimal basic solution and let B be the corresponding optimal basis (which is not unique in general). Denote by $N := \{1,\ldots,n\} \setminus B$ the set of nonbasic indices.

It is well known (Gal and Greenberg, 1997; Ward and Wendell, 1990; Jansen et al., 1997; Gál and Nedoma, 1971) that the critical region of optimal basis invariancy has the description

$$\Upsilon_B(x^*) = \{\lambda \in \mathbb{R}^n | \lambda_N^T - \lambda_B^T A_B^{-1} A_N \geqslant 0\}. \tag{1}$$

The optimal value function is $\lambda^T x^*$ on $\Upsilon_B(x^*)$.

2.2. Support set invariancy

Recall that x^* is any optimal solution of (P_{λ^0}). Let $P := \sigma(x^*)$ and $Z := \{1,\ldots,n\} \setminus P$. We start with the first case of support set invariancy and derive a description of the set $\Upsilon_1(x^*)$.

Theorem 1. *Let $h_i, i \in I$, be a basis of the lineality space $\mathscr{L}_Z := \{x | Ax = 0, x_Z = 0\}$ and let $g_j, j \in J$, be all extremal directions of the convex polyhedral cone $\{x | Ax = 0, x_Z \geqslant 0\} \cap \mathscr{L}_Z^{\perp}$. Then*

$$\Upsilon_1(x^*) = \left\{\lambda \in \mathbb{R}^n | h_i^T \lambda = 0\ \forall i \in I,\ g_j^T \lambda \geqslant 0\ \forall j \in J\right\}. \tag{2}$$

Proof. The tangent cone to the primal feasible set $\mathscr{P} = \{x | Ax = b, x \geqslant 0\}$ at the point x^* is $\mathscr{T}(x^*) = \{x | Ax = 0, x_Z \geqslant 0\}$. The normal (polar) cone to $\mathscr{T}(x^*)$ at the point x^* is described by (see e.g. Nožička et al., 1974, 1988; Rockafellar and Wets, 2004)

$$\mathscr{T}^*(x^*) = \left\{\lambda \in \mathbb{R}^n | h_i^T \lambda = 0\ \forall i \in I,\ g_j^T \lambda \geqslant 0\ \forall j \in J\right\}. \tag{3}$$

We will show that $\mathscr{T}^*(x^*) = \Upsilon_1(x^*)$. If $\lambda \in \mathscr{T}^*(x^*)$ then x^* is an optimal solution to (P_λ), and hence $\lambda \in \Upsilon_1(x^*)$. Conversely, if $\lambda \in \Upsilon_1(x^*)$ then there is an optimal solution x to (P_λ) such that $\sigma(x) = \sigma(x^*)$. The tangent cones at x^* and x are the same, and alike the normal cones, which means that $\lambda \in \mathscr{T}^*(x) = \mathscr{T}^*(x^*)$. □

Notice that if x^* is a nondegenerate optimal basic solution, then $\Upsilon_B(x^*) = \Upsilon_1(x^*)$. Without the condition of nondegeneracy, we have only the inclusion $\Upsilon_B(x^*) \subseteq \Upsilon_1(x^*)$. It is true because $\Upsilon_1(x^*)$ is formed by a union of $\Upsilon_B(x^*)$s for all basis corresponding to the vertex x^*.

A basis of the lineality space $\mathscr{L}_Z$ can be found efficiently. But computation of all extremal directions of the convex polyhedral cone is a hard (exponential) problem; the methods which are used includes simplex algorithm and Nožička method (via constructing a convex basis of a cone, see Nožička et al., 1974). Let us note that $\Upsilon_1(x^*)$ has an alternative description by means of generators (instead of inequalities) as the convex hull by

$$\Upsilon_1(x^*) = \mathrm{cone}\{\pm A_{i\cdot}\ \forall i = 1,\ldots,m,\ e_i\ \forall i \in Z\},$$

where e_i stands for the ith unit vector. Although this characterization of $\Upsilon_1(x^*)$ does not require any additional computations, the formula (2) is much more useful from the practical viewpoint.