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Short Communication

Dynamic mean–variance portfolio selection with borrowing constraint

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ABSTRACT

This paper derives explicit closed form solutions, for the efficient frontier and optimal investment strategy, for the dynamic mean–variance portfolio selection problem under the constraint of a higher borrowing rate. The method used is the Hamilton–Jacobi–Bellman (HJB) equation in a stochastic piecewise linear-quadratic (PLQ) control framework. The results are illustrated on an example.

1. Introduction

Mean–variance portfolio selection has been a central issue in finance since Markowitz's pioneering work [17] on a single-period investment model. Generalizations of this work have followed two main venues. On one hand, multi-period portfolio selection has been extensively studied, see for instance, Mossin [19], Samuelson [21], Hakansson [11], Francis [9], Compbell et al. [2], Li and Ng [14] and the references therein. On the other hand, continuous-time portfolio selection models have been investigated by Merton [18], Karatzas et al. [13], Cox and Huang [5], Duffie and Richardson [7], Dumas and Luciano [8], Grossman and Zhou [10], Zhou and Li [25]. The martingale approach to utility maximization, was independently developed by Karatzas et al. [13], and Cox and Huang [5] for the case of complete markets. The utility function is usually assumed to be a continuous, increasing and strictly concave function such as a power, logarithm, exponential or quadratic function. The risk and return relationship is implicit in the utility function approach and cannot be disentangled at the level of optimal strategies. We note that optimality in the utility theory does not necessarily correspond to optimality under mean–variance, and that these two constitute different approaches. Constrained portfolio selection problems have been extensively studied by Cvitanic and Karatzas [6], Paxson [20], Karatzas and Kou [12], Carassus et al. [1], using mainly the utility function theory, and the duality approach, which was then extended by [6,22].

Recently, using the so-called indefinite stochastic linear-quadratic (LQ) control theory (see, e.g., [3,4,24]), Zhou and Li [25] introduced a continuous-time mean–variance portfolio selection model without any constraints, leading to a closed form analytical optimal portfolio strategy and obtained an explicit expression of the efficient frontier. This work used the classical Riccati approach. Next, [15] considered the same problem but with a short-selling constraint. Since the Riccati approach fails in this case, the Hamilton–Jacobi–Bellman equation was directly used. The novelty in this LQ control approach, as opposed to the duality method of [6,22] was that viscosity solution techniques as in [26] were used.

In this paper, we consider continuous-time mean–variance portfolio selection with a new constraint, that is a borrowing constraint, i.e, under different interest rates for borrowing and lending, rendering the market incomplete. This constraint forces the problem to become piecewise linear-quadratic and is hence no longer LQ. As a result the Riccati approach fails again. We construct two special Riccati

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equations as a continuous (actually a viscosity) solution to the HJB equation. We obtain an explicit closed form solution for the optimal strategy as well as the efficient frontier. In this investment framework, we investigate and quantify how the borrowing restriction affects the relative investments in different risky assets. For ease of readability, the main financial results are presented first, and illustrated by an example next, and the technical mathematical proof is differed to the last section.

The outline of this paper is as follows: Section 2 gathers the notation and problem formulation. Section 3 presents the mean–variance portfolio strategy theorem under the constraint of a higher borrowing rate. Section 4 contains the main theorems on the efficient frontier and the optimal portfolio strategy. Section 5 illustrates the main results on a test case example. Section 6 contains the technical proof of the main theorem in Section 3, in the stochastic piecewise linear-quadratic (PLQ) control framework. Section 7 concludes this work.

2. Problem formulation

Throughout this paper we denote by M' the transpose of any matrix or vector $M = (m_{ij})$, $\|M\| = \sqrt{\sum_{i,j} m_{ij}^2}$ its norm, $\mathbb{R}^n$ the n dimensional real space, $\mathbf{1} = (1, \ldots, 1)$ the vector with unit entries, $x^- = -\min(x, 0)$, $[0, T]$ the finite horizon of investment and E the expectation. Market uncertainty is modeled by a filtered complete probability space $(\Omega, \mathscr{F}, \mathbf{P}, \{\mathscr{F}_t\}_{t \geqslant 0})$ and a standard $\{\mathscr{F}_t\}_{t \geqslant 0}$-adapted m-dimensional Brownian motion $W(t) \equiv (W^1(t), \ldots, W^m(t))'$. We recall that the space $L^2_{\mathscr{F}}(0, T, \mathbb{R}^m)$ of mean square integrable functions $u(t, X)$ from $\mathbb{R} \times \Omega$ to $\mathbb{R}^m$ possesses a natural Hilbert space structure, inducing the norm

$$\|u\|_{\mathscr{F},2} = \left(E \int_0^T \|u(t, X)\|^2 dt \right)^{\frac{1}{2}} < +\infty.$$

We consider a financial market where $m + 1$ assets are traded continuously over $[0, T]$. The first asset is a bond whose price $S_0(t)$ evolves under two different borrowing and lending rates, according to the differential equation

$$\begin{cases} dS_0(t) = \begin{cases} r(t)S_0(t)dt, & \text{if } S_0(t) \geqslant 0, \quad t \in [0, T], \\ R(t)S_0(t)dt, & \text{if } S_0(t) < 0, \quad t \in [0, T], \end{cases} \\ S_0(0) = s_0 > 0, \end{cases} \tag{1}$$

i.e.,

$$\begin{cases} dS_0(t) = (r(t)S_0(t) - (R(t) - r(t))S_0(t)^-)dt, \quad t \in [0, T], \\ S_0(0) = s_0 > 0, \end{cases}$$

where $r(t)(> 0)$ is the interest rate of the bond and $R(t)$ is the borrowing rate being larger than $r(t)$. The remaining m assets are modeled by GBM

$$\begin{cases} dS_i(t) = S_i(t)(b_i(t)dt + \sum_{j=1}^m \sigma_{ij}(t)dW^j(t)), \quad t \in [0, T], \\ S_i(0) = s_i > 0, \end{cases} \tag{2}$$

where $b(t) := (b_1(t), \ldots, b_m(t))'$ are the drifts with $b_i(t) > R(t)$ and $\sigma(t) := (\sigma_{ij}(t))$ is the volatility matrix. We assume throughout that $r(t), R(t)$, $b(t)$ and $\sigma(t)$ are deterministic, Borel-measurable and bounded on $[0, T]$, and that the following non-degeneracy condition holds

$$\sigma(t)\sigma(t)' > 0, \quad \forall t \in [0, T]; \tag{3}$$

that is, $\sigma(t)\sigma(t)'$ is positive semi-definite $\forall t \in [0, T]$. Denote by $X(t)$ an investor's total wealth at time $t \geqslant 0$, invested in $N_i(t)$ shares of the ith asset $(i = 0, 1, \ldots, m)$, that is $X(t) = \sum_{i=0}^m N_i(t)S_i(t), t \geqslant 0$. Let $u_i(t, X(t))$ or $u_i(t)$ for short denote $N_i(t)S_i(t)$, that is the total market value of the investor's wealth in the ith stock and $u_0(t) = N_0(t)S_0(t)$ in the bond. In particular, $X_0 > 0$ is the initial wealth. In this work, transaction cost and consumptions are not considered, and trading is assumed to take place continuously. Since the $N_i(t)$ are integers, or invoking self-financing, the dynamics of the *wealth process* $X(t), t \geqslant 0$ after re-arranging some terms is seen to follow the stochastic differential equation,

$$\begin{aligned} dX(t) &= \sum_{i=0}^m N_i(t)dS_i(t) = (r(t)N_0(t)S_0(t) - (R(t) - r(t))N_0(t)S_0(t)^- + \sum_{i=1}^m b_i(t)N_i(t)S_i(t))dt + \sum_{i=1}^m N_i(t)S_i(t) \sum_{j=1}^m \sigma_{ij}(t)dW^j(t) \\ &= (r(t)X(t) + \sum_{i=1}^m (b_i(t) - r(t))u_i(t) - (R(t) - r(t))(X(t) - \sum_{i=1}^m u_i(t))^-)dt + \sum_{j=1}^m \sum_{i=1}^m \sigma_{ij}(t)u_i(t)dW^j(t), X(0) = X_0. \end{aligned} \tag{4}$$

Note that the portfolio $u(t) := (u_1(t), \ldots, u_m(t))'$ is dynamic i.e., changes over time.

Remark 2.1. We do not constrain the amount $u_i(t)$ invested in the ith stock to take positive values, in other words, short-selling of stocks is allowed.

Remark 2.2. Borrowing from the money market at the rate $R(t)$ is allowed. This is reflected in the term $(R(t) - r(t))(X(t) - \sum_{i=1}^m u_i(t))^-$ of (4). Because of this, this problem is no longer linear-quadratic, and is piecewise linear-quadratic, or (PLQ). See Section 7 for more detail.

Remark 2.3. The above features distinguish this work from [15] and lead to a different mathematical formulation.

For a prescribed target expected terminal wealth $EX(T) = K$, mean–variance portfolio optimization consists of determining a dynamic portfolio satisfying all the constraints of a given framework, and minimizing the risk as measured by the variance of the terminal wealth, that is minimizing

$$\mathrm{Var}X(T) = E[X(T) - EX(T)]^2 = E[X(T) - K]^2.$$