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The variable radius covering problem

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ABSTRACT

In this paper we propose a covering problem where the covering radius of a facility is controlled by the decision-maker; the cost of achieving a certain covering distance is assumed to be a monotonically increasing function of the distance (i.e., it costs more to establish a facility with a greater covering radius). The problem is to cover all demand points at a minimum cost by finding optimal number, locations and coverage radii for the facilities. Both, the planar and discrete versions of the model are considered. Heuristic approaches are suggested for solving large problems in the plane. These methods were tested on a set of planar problems. Mathematical programming formulations are proposed for the discrete problem, and a solution approach is suggested and tested.

1. Introduction

The location set covering problem (LSCP), introduced by ReVelle et al. (1976), is one of the classic models in the location literature. The problem is to cover the customer demand, assumed to be concentrated at a discrete set of points, with the minimum number of facilities. Each facility has a constant coverage radius r, a demand point is assumed to be "covered" if it is within distance r of a facility. This model has a wide range of applications, including emergency response system design, location of retail facilities, design of computer networks, etc. (for reviews see Schilling et al., 1993; Daskin, 1995; Current et al., 2002; Plastria, 2002). A close relative of LSCP is the maximal cover location problem MCLP, where the goal is to cover the maximum amount of demand with a certain number of facilities; for network formulations see Church and ReVelle (1974), Megiddo et al. (1983), ReVelle (1986) and Berman (1994), and for planar problems see Drezner (1981), Watson-Gandy (1982), Drezner (1986) and Canovas and Pelegrin (1992).

The covering radius r in LSCP (and in MCLP) is assumed to be an exogenous parameter, outside of the control of the decision-maker. However, in many (if not most) applications, the coverage radius of a facility is one of the design parameters: a larger or smaller coverage radius can be achieved by increasing or decreasing the "size" of the facilities (here by "size" we refer to the physical characteristic of the facility that is related to the coverage radius). For example, locating warning sirens (Current and O'Kelly, 1992) is modeled as a p-center problem but the coverage radius depends on the intensity of the siren. When locating light posts to illuminate a certain area, the coverage radius of each individual light depends on the intensity of the light bulb and the height of the post. When locating detectors to warn of fire or other hazards (Drezner and Wesolowsky, 1997), the distance at which the detector discovers the hazard depends on the sensitivity of the detector. The signal strength of a radio station determines the coverage area. In the design of cellular telephone networks, the height of the tower and the signal strength of the transmitter affect the coverage radius. Retailers generally expect larger stores to have larger trading areas. In design of public service facilities (such as schools or hospitals), larger facilities serve more patients (or students) leading to a larger covering distance. Longer runways at an airport allow it to service larger planes, thus allowing arrivals and departures of airplanes from larger distances.

The common feature of the examples above is that, at a certain cost, it is possible to adjust the coverage radius of a facility, with larger coverage radii requiring larger capital investments. In this paper we propose a Variable Radius Covering Problem where a decision-maker has to determine the optimal number, locations and coverage radii for the facilities to cover a discrete number of demand points at a minimal cost. We assume that the cost of constructing a facility with coverage radius r is given by a non-decreasing cost function $\phi(r)$. Two versions of the model are analyzed: in the planar version, the facilities can be located anywhere within a certain region of a plane; in the discrete version, the demand is assumed to come from a finite set of points and the facilities locations set is also finite.

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To the best of our knowledge, the direct trade-off between the coverage radius and the facility cost is novel to our model. However, there has been some prior work on relaxing the "fixed coverage radius" assumption of the LSCP. One stream of research investigated the hierarchical covering problem where a set of different possible coverage radii is given and two possible objectives apply (Daskin and Stern, 1981; Church and Eaton, 1987). The maximal expected covering problem (where there is some uncertainty about whether a demand point will be covered) is investigated in Daskin (1983) and Batta et al. (1989). Another stream of research, in the MCLP context, is gradual covering problems, where demand points that are too far away from the facility are assumed to be only partially covered. In one version of the problem (Berman and Krass, 2002) the proportion of the demand covered is a decreasing stepwise function of the coverage radius. In another version there is a minimum and maximum covering distance. The demand is fully covered within the minimum distance and is not covered at all beyond the maximum distance. Between these two distances the coverage is gradually declining either linearly or otherwise (Berman et al., 2003; Drezner et al., 2004). In Drezner et al. (2004) the single facility case in the plane using Euclidean distances is optimally solved.

Carrizosa and Plastria (1998) and Plastria and Carrizosa (1999) studied covering problems with varying radii. In their models the coverage radius is the same for all facilities. They develop the efficient frontier between the covering radius and the maximal cover with that radius. We note that in our model, each facility will, in general, have its own coverage radius.

The idea of selecting the attractiveness of a facility as a variable in competitive location models was suggested in Drezner (1998), Plastria and Carrizosa (2004), Fernandez et al. (2007) and Aboolian et al. (2008). This is similar, in spirit, to our model in which the cost of building a facility is a function of its coverage radius (attractiveness). However, the objective of our model and of the competitive location models mentioned above are obviously quite different: in our mode, it is to cover all demand point at the minimal total cost; in competitive location models it is generally to select the most profitable strategy, given certain actions of the competitor(s).

This paper is organized as follows. In Section 2 we develop the model in the plane, establish certain structural results, and construct three heuristic approaches. In Section 3 we report computational experience with these heuristic algorithms; two of the heuristics appear to be quite efficient in obtaining accurate solutions for problems with up to 10,000 demand nodes. The experiments are performed on both, randomly generated and real-life problem sets. In Section 4 we formulate the discrete model, and develop an exact solution algorithm. In Section 5 we report computational experiments with a set of test problems on a network; the algorithm appears to be capable of handling instances with up to 1000 demand points in a few seconds. We conclude in Section 6 and suggest topics for future research.

2. The planar Variable Radius Covering Problem

In the plane, each facility covers all the demand points within a circle of a given radius. While the model could be developed with any distance metric, we will use Euclidean distances, unless stated otherwise.

We introduce the following notation:

Decision variables

- p the unknown number of facilities
- $X_j = (x_j, y_j)$ be the unknown location of facility j for $j = 1, \dots, p$
- X the vector $\{X_j\}$ for $j = 1, \dots, p$
- Y_{ij} a binary variable. $Y_{ij} = 1$ if demand point i is assigned to facility j, and $Y_{ij} = 0$ otherwise
- Y the matrix $\{Y_{ij}\}$
- r_j the unknown coverage radius for facility $j = 1, \dots, p$
- $P = \{p, X, (r_1, \dots, r_p)\}$ the facility set describing the number, locations, and coverage radii of the facilities

Problem parameters

- n the number of demand points
- (a_i, b_i) the location of demand point i for $i = 1, \dots, n$
- $d_i(X_j)$ the Euclidean distance between demand point i and facility j
- F the non-negative fixed cost of locating one facility
- $\phi(r)$ the variable cost of building a facility of radius r. By definition $\phi(0) = 0$

The total cost of building a facility with coverage radius $r > 0$ is assumed to be $F * I\{r > 0\} + \phi(r)$, where $I\{\cdot\}$ is the indicator function (for $r = 0$ the total cost is zero and the fixed cost is not charged). The variable cost function $\phi(r)$ is assumed to be non-decreasing in r. Note that to minimize the overall cost of coverage, for a given variable cost expenditure, it is always beneficial to construct a facility with the largest possible coverage radius. Thus, even if a facility with the larger radius has a lower cost than one with a smaller radius, the larger radius option will always be chosen, thus the assumption above is made without loss of generality. We do not assume continuity of $\phi(r)$ as construction costs may jump in value at some specific radii.

An example of the cost function applicable in several settings is $\phi(r) = Cr^2$ for $C > 0$: when coverage is defined as a physical intensity of the transmitter/receiver, such intensity is often inversely proportional to the square of the distance. Note that if the total number of facilities p is fixed, rather than being a decision variable, we can take $F = 0$ for the fixed cost component (this is the case in many of the competitive location models mentioned in Section 1). Several different forms of the coverage cost function have been suggested by other authors. For example, in Drezner (1998) four cost structures were suggested: a decreasing marginal return curve $\phi(r) = C\sqrt{r}$, a fixed marginal return curve $\phi(r) = Cr$, and two increasing marginal return curves, $\phi(r) = Cr^2$ for rapidly increasing, and $\phi(r) = Cr(1 + \alpha r)$ for mildly increasing ($\alpha > 0$). In Plastria and Carrizosa (2004) a list of possible radii each with a corresponding cost is suggested. In Fernandez et al. (2007) it is required that $\phi(r)$ is differentiable and the function may be different for different demand points. They suggest $\phi(r) = Cr^k$ for $k \geqslant 1$ or $\phi(r) = C(e^{kr} - 1)$ for $k > 0$.

Since the variable cost is non-decreasing in the coverage radius, the latter is uniquely determined by the assignment variables through the following relationship:

$$r_j = \max_{1 \leqslant i \leqslant n}\{Y_{ij} d_i(X_j)\}$$

(i.e., each facility must cover all demand assigned to it and the coverage radius is determined by the furthest assigned demand point). It follows that the cost corresponding to given values of the decision variables X, Y and p is

$$F(X, Y, p) = pF + \sum_{j=1}^{p} \phi\left(\max_{1 \leqslant i \leqslant n}\{Y_{ij} d_i(X_j)\}\right),$$

and the problem can be stated as follows:

$$\min_{X,Y,p}\left\{F(X, Y, p) | Y_{ij} \in \{0, 1\}, i = 1, \dots, n, j = 1, \dots, p; \sum_{j=1}^{p} Y_{ij} = 1\right\}. \tag{1}$$

We note that the formulation (1) above cannot be directly solved by standard mathematical programming solvers since the number of decision variables depends on p, which is itself as a variable. For this reason we develop several heuristic algorithms developed for