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Cagan type rational expectation model on complex discrete time domains

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ABSTRACT

In this article, we derive a solution for a linear stochastic model on a complex time domain. In this type of models, the time domain can be any collection of points along the real number line, so these models are suitable for problems where events do not occur at evenly-spaced time intervals. We present examples based on well-known results from economics and finance to illustrate how our model generalizes and extends conventional dynamic models.

1. Introduction

In many business disciplines, especially in economics and finance, researchers are interested in modeling dynamics of variables whose current values are influenced by their own expected future values. Perhaps the best known example of such a model is the Cagan (1956) model of hyperinflation. In that model, the current price level depends on how economic agents form expectations about the future price level. In finance, the current price of an asset can be modeled as a function of a future price of that asset. Generally, such models are expressed as

$$y_t = aE[y_{t+1}|I_t] + cz_t, \tag{1.1}$$

where variable y_t is a linear function of its own expected future value, conditional on I_t, the information set available at time t, and an exogenous variable z_t, which can be either a deterministic or a random variable.

This type of model served as a building block for many seminal works. Lucas (1973) employed this type of model to study output-inflation tradeoff. Sargent and Wallace (1975) used it to study the effectiveness of various monetary policies. Sargent (1977) and Taylor (1979) studied econometric methods for estimating a dynamic model with expectations. These works and many others shaped modern macroeconomics.

Theoretical and empirical aspects of this model continue to receive attention in the literature. Blanchard and Kahn (1980), Broze, Janssen, and Szafarz (1984) and Gourieroux, Laffont, and Monfort (1982) analyzed solutions to general versions of this model which can include expectations of multiple future values and lags of y. Christiano (1987) discussed econometric issues pertaining to this type of model.

In this paper, we generalize the model in (1.1) by allowing the time period between the current period and the period for which expectation is formed to be of any arbitrary length. A related work by Tucci (2004) takes a step in this direction by restating Cagan's model using m-period future expectations:

$$y_t = aE_t[y_{t+m}] + z_t, \tag{1.2}$$

where $m > 1$, so y is a function of not just a one-period-ahead expectation but an expectation at some arbitrary point in the future.

The model we propose in this paper is

$$y_t = aE_t\left[y_t^\sigma\right] + f(t, z_t). \tag{1.3}$$

The principal difference between the models in (1.1) and (1.2) is in how they approach timing of events. The time domain of (1.1) is a set of integers, so events in this model occur at evenly spaced time intervals. The time domain of (1.3) can be any collection of points on $\mathbb{R}^+$, the set of positive real numbers, and y_t and y_t^σ are values of y on two consecutive points on the time domain. Therefore, in (1.3), y_t can be a function of its expected value at any arbitrary point in the future, and the interval between point t and the next point on the time domain can vary over time. The models in (1.1) and (1.2) are special cases of the model in (1.3). Thus, the model in (1.3) is a generalization of existing models in the literature.

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In the next section, we present the theory behind the dynamic model used in this paper and derive the solution for the model in (1.3). Then we discuss two examples to draw parallels with existing models and to show how the model proposed in this paper contributes to the literature on dynamic modeling.

2. Dynamic equations on complex discrete time domains

Let $\mathbb{T} = \{0 = t_0, t_1, t_2, t_3, \ldots\}$ be the set of positive real numbers such that $t_i < t_j$ for $i < j$. Then we define $\sigma(t_i) = t_{i+1}$ and $\mu(t_i) = \sigma(t_i) - t_i$ for $t_i \in \mathbb{T}$. With these properties, the time domain $\mathbb{T}$ can be considered as a complex discrete time domain.

Let f be a real valued function defined on $\mathbb{T}$. Then the Δ-derivative of f is defined as

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)},$$

where $t \in \mathbb{T}$. The Δ-integral of f is defined as

$$\int_0^T f(\tau)\Delta\tau = \sum_{s\in[0,T)\cap\mathbb{T}} \mu(s) f(s),$$

where $0, T \in \mathbb{T}$.

The exponential function on $\mathbb{T}$ is denoted as

$$e_p(t,t_0) = \prod_{s\in[t_0,t)\cap\mathbb{T}} (1 + \mu(s)p(s)),$$

where $1 + p(t)\mu(t) \neq 0$ for all $t \in \mathbb{T}$.

If $\mathbb{T} := \mathbb{Z}$ and p is a constant so that $1 + p\mu(t) \neq 0$ for all $t \in \mathbb{Z}$, then $e_p(t,t_0) = (1+p)^{t-t_0}$. The circle minus of $p(t)$, $\ominus p(t)$, is defined to be $-\frac{p(t)}{1+\mu(t)p(t)}$ for all $t \in \mathbb{T}^{\kappa}$. If $1 + p\mu(t) > 0$, then $e_p(t,t_0) > 0$ for all $t \in \mathbb{T}$ (for details, see Merrell, Ruger, & Severs, 2004).

With the help of these definitions we derive the general solution for the model (1.3) where f may be a linear or a nonlinear function. We then illustrate how our model and its solution generalize the existing results in the literature (Broze, Gourieroux, & Szafarz, 1985; Broze et al., 1984; Broze & Szafarz, 1984; Gourieroux et al., 1982; Pesaran, 1981). In the statement of the next theorem, we use the notation given in the paper (Merrell et al., 2004), and we drop the variable t of $\mu(t)$, which appears in the subscripts of $e_{\frac{1-a}{a\mu(t)}}$ and $e_{\ominus\frac{1-a}{a\mu(t)}}$.

Theorem 2.1. *Let $\mathbb{T}$ be a complex discrete time domain. Then the following y_t solves the equation in (1.3)*

$$y_t = e_{\frac{1-a}{a\mu}}(t,0)M(t) - e_{\frac{1-a}{a\mu}}(t,0)\int_0^t e_{\ominus\frac{1-a}{a\mu}}(s,0)\frac{1}{\mu(s)}f(s,z_s)\Delta s, \tag{2.1}$$

where $t \in \mathbb{T}$ and $M(t)$ is an arbitrary martingale on $\mathbb{T}$, i.e. it satisfies the martingale property

$$E_t[M^{\sigma}(t)] = M(t).$$

Proof. We start by shifting y_t one step forward. Hence, we have

$$y_t^{\sigma} = e_{\frac{1-a}{a\mu}}(\sigma(t),0)M^{\sigma}(t) - e_{\frac{1-a}{a\mu}}(\sigma(t),0)\int_0^{\sigma(t)} e_{\ominus\frac{1-a}{a\mu}}(s,0)\frac{1}{\mu(s)}f(s,z_s)\Delta s.$$

Since $e^{\sigma}_{\frac{1-a}{a\mu}}(t,0) = \frac{1}{a}e_{\frac{1-a}{a\mu}}(t,0)$, it follows that

$$y_t^{\sigma} = \frac{1}{a}e_{\frac{1-a}{a\mu}}(t,0)M^{\sigma}(t) - \frac{1}{a}e_{\frac{1-a}{a\mu}}(t,0)\int_0^{\sigma(t)} e_{\ominus\frac{1-a}{a\mu}}(s,0) \times \frac{1}{\mu(s)}f(s,z_s)\Delta s. \tag{2.2}$$

Using properties of the integral, we write the above expression (2.2) as

$$y_t^{\sigma} = \frac{1}{a}e_{\frac{1-a}{a\mu}}(t,0)M^{\sigma}(t) - \frac{1}{a}e_{\frac{1-a}{a\mu}}(t,0) \times \left\{\int_0^t e_{\ominus\frac{1-a}{a\mu}}(s,0)\frac{1}{\mu(s)}f(s,z_s)\Delta s + \int_t^{\sigma(t)} e_{\ominus\frac{1-a}{a\mu}}(s,0)\frac{1}{\mu(s)}f(s,z_s)\Delta s\right\}.$$

Because $\int_t^{\sigma(t)} e_{\ominus\frac{1-a}{a\mu}}(s,0)\frac{1}{\mu(s)}f(s,z_s)\Delta s = e_{\ominus\frac{1-a}{a\mu}}(t,0)f(t,z_t)$, we obtain

$$y_t^{\sigma} = \frac{1}{a}e_{\frac{1-a}{a\mu}}(t,0)M^{\sigma}(t) - \frac{1}{a}e_{\frac{1-a}{a\mu}}(t,0)\left\{\int_0^t e_{\ominus\frac{1-a}{a\mu}}(s,0)\frac{1}{\mu(s)}f(s,z_s)\Delta s + e_{\ominus\frac{1-a}{a\mu}}(t,0)f(t,z_t)\right\}. \tag{2.3}$$

Next, multiplying each side of the above Eq. (2.3) by a and then applying the conditional expectation to each side, we obtain

$$aE_t y_t^{\sigma} = e_{\frac{1-a}{a\mu}}(t,0)E_t M^{\sigma}(t) - e_{\frac{1-a}{a\mu}}(t,0)\left\{\int_0^t e_{\ominus\frac{1-a}{a\mu}}(s,0)\frac{1}{\mu(s)}f(s,z_s)\Delta s + e_{\ominus\frac{1-a}{a\mu}}(t,0)f(t,z_t)\right\}. \tag{2.4}$$

As a last step, we subtract $aE_t y_t^{\sigma}$ in (2.4) from y_t in (2.1) side by side and we obtain

$$y_t - aE_t y_t^{\sigma} = e_{\frac{1-a}{a\mu}}(t,0)\{M_t - E_t M_t^{\sigma}\} + e_{\frac{1-a}{a\mu}}(t,0)e_{\ominus\frac{1-a}{a\mu}}(t,0)f(t,z_t).$$

Because $e_{\frac{1-a}{a\mu}}(t,0)e_{\ominus\frac{1-a}{a\mu}}(t,0) = 1$, we have

$$y_t - aE_t y_t^{\sigma} = f(t,z_t).$$

This completes the proof. □

Corollary 2.1. *If $\mathbb{T} = \mathbb{Z}$, then the Eq. (2.1) becomes*

$$y_t = \frac{1}{a^t}M_t - \frac{1}{a^t}\sum_{s=0}^{t-1} a^s f(s,z_s), \tag{2.5}$$

where M_t is a discrete time martingale.

Remark 2.1. If $f \equiv 0$, Eq. (1.3) is reduced to

$$y_t = aE_t[y_t^{\sigma}],$$

and the solution for y is

$$y_t = \frac{1}{a^t}M_t.$$

Remark 2.2. Broze et al. (1985) show that the general solution to the model in (1.3) on $\mathbb{T} = \mathbb{Z}$ is

$$y_t = \frac{1}{a}y_{t-1} + \varepsilon_t^0 - \frac{1}{a}f(t-1, z_{t-1}), \tag{2.6}$$

where ε_t^0 is an arbitrary martingale difference.

Solution of the Eq. (2.6) coincides with the solution (2.1) for $\mathbb{T} = \mathbb{Z}$. To see this, first consider the case where $f \equiv 0$. Then the Eq. (2.6) becomes

$$y_t = \frac{1}{a}y_{t-1} + \varepsilon_t^0. \tag{2.7}$$

By shifting the above Eq. (2.7) one unit forward and multiplying each side of the equation by a^{t+1} we have

$$a^{t+1}y_{t+1} - a^t y_t = a^{t+1}\varepsilon_{t+1}^0. \tag{2.8}$$

Next, we sum each side of Eq. (2.8) from 0 to $t-1$ and now we have