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Decision Support

The uniqueness property for networks with several origin–destination pairs

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ABSTRACT

We consider congestion games on networks with nonatomic users and user-specific costs. We are interested in the uniqueness property defined by Milchtaich (2005) as the uniqueness of equilibrium flows for all assignments of strictly increasing cost functions. He settled the case with two-terminal networks. As a corollary of his result, it is possible to prove that some other networks have the uniqueness property as well by adding common fictitious origin and destination. In the present work, we find a necessary condition for networks with several origin–destination pairs to have the uniqueness property in terms of excluded minors or subgraphs. As a key result, we characterize completely bidirectional rings for which the uniqueness property holds: it holds precisely for nine networks and those obtained from them by elementary operations. For other bidirectional rings, we exhibit affine cost functions yielding to two distinct equilibrium flows. Related results are also proven. For instance, we characterize networks having the uniqueness property for any choice of origin–destination pairs.

1. Introduction

In many areas, different users share a common network to travel or to exchange information or goods. Each user wishes to select a path connecting a certain origin to a certain destination. However, the selection of paths in the network by the users induces congestion on the arcs, leading to an increase of the costs. Taking into account the choices of the other users, each user looks for a path of minimum cost. We expect therefore to reach a Nash equilibrium: each user makes the best reply to the actions chosen by the other users.

This kind of games is studied since the 1950s, with the seminal works by Wardrop (1952) and Beckmann, McGuire, and Winsten (1956). Their practical interest is high since the phenomena implied by the strategic interactions of users on a network are often nonintuitive and may lead to an important loss in efficiency. The Braess paradox (Braess, 1968) – adding an arc may deteriorate all travel times – is the classical example illustrating such a nonintuitive loss and it has been observed in concrete situations, for example in New York Kolata (1990). Koutsoupias and Papadimitriou (1999) initiated a precise quantitative study of this loss, which lead soon after to the notion of "Price of Anarchy" that is the cost of the worst equilibrium divided by the optimal cost, see (Roughgarden & Tardos, 2002) among many other references. Some recent researches proposed also ways to control this loss, see Bauso, Giarré, and Pesenti (2009) and Knight and Harper (2013) for instance.

When the users are assumed to be nonatomic – the effect of a single user is negligible – equilibrium is known to exist (Milchtaich, 2000). Moreover, when the users are affected equally by the congestion on the arcs, the costs supported by the users are the same in all equilibria (Aashtiani & Magnanti, 1981). In the present paper, we are interested in the case when the users may be affected differently by the congestion. In such a case, examples are known for which these costs are not unique. Various conditions have been found that ensure nevertheless uniqueness. For instance, if the user's cost functions attached to the arcs are continuous, strictly increasing, and identical up to additive constants, then we have uniqueness of the equilibrium flows, and thus of the equilibrium costs (Altman & Kameda, 2001). In 2005, continuing a work initiated by Milchtaich (2000) and Konishi (2004) for networks with parallel routes, Milchtaich (2005) found a topological characterization of two-terminal networks for which, given any assignment of strictly increasing and continuous cost functions, the flows are the same in all equilibria. Such networks are said to enjoy the *uniqueness property*. Similar results with atomic users have been obtained by Orda, Rom, and Shimkin (1993) and Richman and Shimkin (2007).

The purpose of this paper is to find similar characterizations for networks with more than two terminals. We are able to characterize completely the ring networks having the uniqueness property,

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whatever the number of terminals is. Studying equilibria on rings can be seen as the decentralized counterpart of works on the optimization of multiflows on rings, like the one proposed by Myung, Kim, and Tcha (1997). Our main result for ring networks is that the uniqueness property holds precisely for nine networks and those obtained from them by elementary operations. For other rings, we exhibit affine cost functions yielding to two distinct equilibrium flows. It allows to describe infinite families of graphs for which the uniqueness property does not hold. For instance, there is a family of ring networks such that every network with a minor in this family does not have the uniqueness property.

2. Preliminaries on graphs

An *undirected graph* is a pair $G = (V, E)$ where V is a finite set of *vertices* and E is a family of unordered pairs of vertices called *edges*. A *directed graph*, or *digraph* for short, is a pair $D = (V, A)$ where V is a finite set of *vertices* and A is a family of ordered pairs of vertices called *arcs*. A *mixed graph* is a graph having edges and arcs. More formally, it is a triple $M = (V, E, A)$ where V is a finite set of vertices, E is a family of unordered pairs of vertices (edges) and A is a family of ordered pairs of vertices (arcs). Given an undirected graph $G = (V, E)$, we define the *directed version* of G as the digraph $D = (V, A)$ obtained by replacing each (undirected) edge in E by two (directed) arcs, one in each direction. An arc of G is understood as an arc of its directed version. In these graphs, *loops* – edges or arcs having identical endpoints – are not allowed, but pairs of vertices occurring more than once – *parallel edges* or *parallel arcs* – are allowed.

A *walk* in a directed graph D is a sequence

$$P = (v_0, a_1, v_1, \dots, a_k, v_k)$$

where $k \geqslant 0, v_0, v_1, \dots, v_k \in V, a_1, \dots, a_k \in A$, and $a_i = (v_{i-1}, v_i)$ for $i = 1, \dots, k$. If all v_i are distinct, the walk is called a *path*. If no confusion may arise, we identify sometimes a path P with the set of its vertices or with the set of its arcs, allowing to use the notation $v \in P$ (resp. $a \in P$) if a vertex v (resp. an arc a) occurs in P.

An undirected graph $G' = (V', E')$ is a *subgraph* of an undirected graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. An undirected graph G' is a *minor* of an undirected graph G if G' is obtained by contracting edges (possibly none) of a subgraph of G. *Contracting* an edge uv means deleting it and identifying both endpoints u and v. Two undirected graphs are *homeomorphic* if they arise from the same undirected graph by subdivision of edges, where a *subdivision* of an edge uv consists in introducing a new vertex w and in replacing the edge uv by two new edges uw and wv.

The same notions hold for directed graphs and for mixed graphs.

Finally, let $G = (V, E)$ be an undirected graph, and $H = (T, L)$ be a directed graph with $T \subseteq V$, then $G + H$ denotes the mixed graph (V, E, L).

3. Model

Similarly as in the multiflow theory (see for instance Schrijver (2003) or Korte & Vygen (2000)), we are given a *supply graph* $G = (V, E)$ and a *demand digraph* $H = (T, L)$ with $T \subseteq V$. The graph G models the (transportation) *network*. The arcs of H model the origin–destination pairs, also called in the sequel the *OD-pairs*. H is therefore assumed to be simple, i.e. contains no loops and no multiple edges. A *route* is an (o, d)-path of the directed version of G with $(o, d) \in L$ and is called an (o, d)-route. The set of all routes (resp. (o, d)-routes) is denoted by $\mathcal{R}$ (resp. $\mathcal{R}_{(o,d)}$).

The population of *users* is modeled as a bounded real interval I endowed with the Lebesgue measure λ, the *population measure*. The set I is partitioned into measurable subsets $I_{(o,d)}$ with $(o, d) \in L$, modeling the users wishing to select an (o, d)-route.

For a given pair of supply graph and demand digraph, and a given partition of users, we define a *strategy profile* as a measurable mapping $\sigma : I \to \mathcal{R}$ such that $\sigma(i) \in \mathcal{R}_{(o,d)}$ for all $(o, d) \in L$ and $i \in I_{(o,d)}$. For each arc $a \in A$ of the directed version of G, the measure of the set of all users i such that a is in $\sigma(i)$ is the *flow* on a in σ and is denoted f_a:

$$f_a = \lambda\{i \in I : a \in \sigma(i)\}.$$

The cost of each arc $a \in A$ for each user $i \in I$ is given by a nonnegative, continuous, and strictly increasing *cost function* $c^i_a : \mathbb{R}_+ \to \mathbb{R}_+$, such that $i \mapsto c^i_a(x)$ is measurable for all $a \in A$ and $x \in \mathbb{R}_+$. When the flow on a is f_a, the cost for user i of traversing a is $c^i_a(f_a)$. For user i, the cost of a route r is defined as the sum of the costs of the arcs contained in r. A *class* is a set of users having the same cost functions on all arcs, but not necessarily sharing the same OD-pair.

The game we are interested in is defined by the supply graph G, the demand digraph H, the population user set I with its partition, and the cost functions c^i_a for $a \in A$ and $i \in I$. If we forget the graph structure, we get a game for which we use the terminology *nonatomic congestion game with user-specific cost functions*, as in Milchtaich (1996).

A strategy profile is a (pure) Nash equilibrium if each route is only chosen by users for whom it is a minimal-cost route. In other words, a strategy profile σ is a Nash equilibrium if for each pair $(o, d) \in L$ and each user $i \in I_{(o,d)}$ we have

$$\sum_{a \in \sigma(i)} c^i_a(f_a) = \min_{r \in \mathcal{R}_{(o,d)}} \sum_{a \in r} c^i_a(f_a).$$

Under the conditions stated above on the cost functions, a Nash equilibrium is always known to exist. It can be proven similarly as Theorem 3.1 in Milchtaich (2000), or as noted by Milchtaich (2005), it can be deduced from more general results (Theorem 1 of Schmeidler (1970) or Theorems 1 and 2 of Rath (1970)). However, such an equilibrium is not necessarily unique, and even the equilibrium flows are not necessarily unique.

4. Results

Milchtaich (2005) raised the question whether it is possible to characterize networks having the *uniqueness property*, i.e. networks for which flows at equilibrium are unique. A pair (G, H) defined as in Section 3 is said to have the *uniqueness property* if, for any partition of I into measurable subsets $I_{(o,d)}$ with $(o, d) \in L$, and for any assignment of (strictly increasing) cost functions, the flow on each arc is the same in all equilibria.

Milchtaich found a positive answer for the two-terminal networks, i.e. when $|L| = 1$. More precisely, he gave a (polynomial) characterization of a family of two-terminal undirected graphs such that, for the directed versions of this family and for any assignment of (strictly increasing) cost functions, the flow on each arc is the same in all equilibria. For two-terminal undirected graphs outside this family, he gave explicit cost functions for which equilibria with different flows on some arcs exist.

The objective of this paper is to address the uniqueness property for networks having more than two terminals. We settle the case of ring networks and find a necessary condition for general networks to have the uniqueness property in terms of excluded minors or subgraphs.

In a ring network, each user has exactly two possible strategies. See Fig. 1 for an illustration of this kind of supply graph G, demand digraph H, and mixed graph $G + H$. We prove the following theorem in Section 5.