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Optimal hedging when the underlying asset follows a regime-switching Markov process ☆

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ABSTRACT

We develop a flexible discrete-time hedging methodology that minimizes the expected value of any desired penalty function of the hedging error within a general regime-switching framework. A numerical algorithm based on backward recursion allows for the sequential construction of an optimal hedging strategy. Numerical experiments comparing this and other methodologies show a relative expected penalty reduction ranging between 0.9% and 12.6% with respect to the best benchmark.

1. Introduction and literature review

For a derivatives trading and risk management activity to be sustainable, hedging is paramount. In practice, portfolio rebalancing is performed in discrete time and the market is typically incomplete, implying that most contingent claims cannot be replicated exactly. Thus, to implement a hedging policy, the challenge is twofold: a model must be specified and hedging strategy objectives must be set.

From a modeling perspective, this article adopts a regime-switching environment. One widely studied class of regime-switching models views log-returns as a mixture of Gaussian variables. These models, introduced in finance by Hamilton (1989), have been shown to improve the statistical fit and forecasts of financial returns. They reproduce widely documented empirical properties such as heteroskedasticity, autocorrelation and fat tails. In this framework, the option pricing problem must deal with incomplete markets and requires the specification of a risk premium. Among significant contributions, Bollen (1998) presents a lattice algorithm to compute the value of European and American options. Hardy (2001) finds a closed-form formula for the price of European options. The continuous-time version of the Gaussian mixture model is studied by Mamon and Rodrigo (2005) who find an explicit value for European options by solving a partial differential equation. Elliott, Chan, and Siu (2005) price derivatives by means of the Esscher transform under the same continuous-time model. Buffington and Elliott (2002) derive an approximate formula for American option prices. Beyond the Gaussian mixture models, extensions address GARCH effects (Duan, Popova, & Ritchken, 2002) and jumps (Lee, 2009a), for example.

Several authors study the problem of hedging an underlying asset with its futures under regime-switching frameworks. Alizadeh and Nomikos (2004) and Alizadeh, Nomikos, and Pouliasis (2008) base their hedging strategy on minimal variance hedge ratios. Lee, Yoder, Mittelhammer, and McCluskey (2006), Lee and Yoder (2007), and Lee (2009a, 2009b) extend the dynamics of the underlying asset in Alizadeh and Nomikos (2004) to incorporate a time-varying correlation between the spot and futures returns, GARCH-type feedback from returns on the volatilty, jumps and copulas for the dependence between futures and spot returns. Lien (2012) provides conditions under which minimal variance ratios taking into account the existence of regimes overperform their unconditional counterparts.

Option hedging under regime-switching models has recently raised interest in the literature. Rémillard and Rubenthaler (2009) adapt the work of Schweizer (1995) to a regime-switching framework and identify the hedging strategy that minimizes the squared error of hedging in both discrete-time and continuous-time for European options. The implementation of this methodology is presented in Rémillard, Hocquard, and Papageorgiou (2010).

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Rémillard, Hocquard, Langlois, and Papageorgiou (2012) extend the hedging procedure to American options.

Another strand of literature discusses self-financing hedging policies[1] under general model assumptions. A widely known methodology is delta hedging. It consists in building a portfolio whose value variations mimick those of the hedged contingent claim when small changes in the underlying asset's value occur. In continuous-time complete markets, delta hedging is the cornerstone of any hedging strategy since it allows for perfect replication. Based on the first derivative of the option price with respect to the underlying asset price, it requires a full characterization of the risk-neutral measure. Many authors discuss the implementation of delta hedging in discrete-time and/or incomplete markets (Duan, 1995, among others). It should be stressed, however, that delta hedging is subject to model misspecification. Nevertheless, it stands as a relevant benchmark when it comes to assessing the performance of a hedging strategy.

Another approach is super-replication (e.g. El Karoui & Quenez, 1995; Karatzas, 1997). It identifies the cheapest trading strategy whose terminal wealth is at least equal to the derivative's payoff. Since the option buyer alone carries the price of the hedging risk, the initial capital required is often unacceptably large. Eberlein and Jacod (1997) show that, under many models, the initial capital required to super-replicate a call option is the price of the underlying asset itself.

An alternative to super-replication is Global Hedging Risk Minimization (GHRM), which consists in identifying trading strategies that replicate the derivative's payoff as closely as possible, or alternatively, minimize the risk associated with terminal hedging shortfalls. Xu (2006) proposes to minimize general risk measures applied to hedging errors. Several authors choose more specific risk measures: quantiles of the hedging shortfall (Cvitanić & Spivak, 1999; Föllmer & Leukert, 1999), expected hedging shortfall (Cvitanić & Karatzas, 1999), expected powers of the hedging shortfall (Pham, 2000), Tail Value-at-Risk (Sekine, 2004), expected squared hedging error (Cont, Tankov, & Voltchkova, 2007; Motoczyński, 2000; Rémillard & Rubenthaler, 2009; Schweizer, 1995) and the expectation of general loss functions (Föllmer & Leukert, 2000). Theoretical existence of optimal hedging strategies under those risk measures and their characterization are studied in a general context. However, explicit solutions exist only for some particular cases of market setups and risk measures. The implementation of the preceding methodologies in the case of incomplete markets is often not straightforward, and tractable algorithms computing the optimal strategies have yet to be identified. The presence of regimes adds an additional layer of difficulty in applying those methods.

This paper's contributions are twofold. First, on a theoretical level, we develop a discrete-time hedging methodology with the GHRM objective that minimizes the expected value of any desired penalty function of the hedging error within a general regime-switching framework (possibly including time-inhomogeneous regime shifts). This methodology is highly flexible and generalizes the quadratic hedging approach. It incorporates a large class of penalty functions encompassing usual risk measures such as Value-at-Risk and expected shortfall. The proposed framework can accommodate portfolio restrictions such as no short-selling. Portfolios can be rebalanced more frequently than the regime-switch timeframe. Second, from an implementation perspective, a numerical algorithm based on backward recursion allows for the sequential construction of an optimal hedging strategy. Numerical experiments challenge our model with existing methodologies. The relative expected penalty reduction obtained with this paper's optimal hedging approach, in comparison with the best benchmark, ranges between 0.9% and 12.6% in the different cases exposed.

This paper is organized as follows. In Section 2, the market model and the hedging problem are described. In Section 3, the hedging problem is solved. Section 4 presents a numerical scheme to compute the solution to the hedging problem. Section 5 presents the market model used for the simulations and provides numerical results. Section 6 concludes the paper.

2. Market specifications and hedging

2.1. Description of the market

Transactions take place in a discrete-time, arbitrage-free financial market. Denote by Δ_t the constant time elapsing between two consecutive observations. Two types of assets are traded. The risk-free asset is a position in the money market account with a nominal amount normalized to one monetary unit. The time – n price of the risk-free asset is

$$S_n^{(1)} = \exp(rn\Delta_t), \quad n \in \{0, 1, 2, \ldots\}$$

where r is the annualized risk-free rate. The price of the risky asset, starting at $S_0^{(2)}$, evolves according to

$$S_n^{(2)} = S_0^{(2)} \exp(Y_n),$$

where Y_n is the risky asset's cumulative return over the time interval $[0, n]$. $\vec{S}_n$ denotes the column vector $\left(S_n^{(1)}, S_n^{(2)}\right)^\top$ and $\vec{S}_{0:n}$ stands for the whole price process up to time n.

The financial market is subject to various regimes that affect the dynamics of the risky asset's price. These regimes are represented by an integer-valued process $\{h_n\}_{n=0}^N$ taking values in $\mathcal{H} = \{1, 2, \ldots, H\}$ where h_n is the regime prevailing during time interval $]n, n+1]$. The joint process (Y, h) has the Markov property[2] with respect to the filtration $\{\mathcal{F}_n\}_{n=0}^N$ satisfying the usual conditions, where

$$\mathcal{F}_n = \sigma(\vec{S}_{0:n}, h_{0:n}) = \sigma(Y_{0:n}, h_{0:n}),$$

meaning that the distribution of (Y_{n+1}, h_{n+1}) conditional on information $\mathcal{F}_n$ is entirely determined by Y_n and h_n.[3] This assumption is consistent with Hamilton (1989) and Duan et al. (2002), among others. Transition probabilities of the regime process h are denoted by

$$P_{i,j}^{(n)}(y) = \mathbb{P}(h_{n+1} = j | h_n = i, Y_n = y) \quad i, j \in \mathcal{H}.$$

Because regimes h are not observable, a coarser filtration $\{\mathcal{G}_n\}_{n=0}^N$ modeling the information available to investors is required, that is, $\mathcal{G}_n = \sigma(Y_{0:n})$.

2.2. The hedging problem

A market participant (referred to as the "hedger") wishes to replicate (or "hedge") the payoff $\phi(S_N^{(2)})$ of a European contingent claim written on the risky asset and maturing at time N, where $\phi(\cdot)$ is some positive Borel function $\phi : [0, \infty) \to \mathbb{R}$. Alternatively, the payoff can be written as a function of the risky asset return

$$\phi(S_N^{(2)}) = \tilde{\phi}(Y_N),$$

for some function $\tilde{\phi}(\cdot)$.

[1] By contrast, local risk-minimization, which considers hedging strategies that are not self-financing, selects one that minimizes a measure of the costs related to non-initial investments in the portfolio (Schweizer, 1991).

[2] A stochastic process $\{X_n\}$ has the Markov property with respect to filtration $\mathcal{F}$ if $\forall n, x$,

$$\mathbb{P}(X_{n+1} \leqslant x | \mathcal{F}_n) = \mathbb{P}(X_{n+1} \leqslant x | X_n).$$

[3] Equivalently, the process $(\vec{S}, h)$ has the Markov property with respect to filtration $\mathcal{F}$.