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# Generating facets for finite master cyclic group polyhedra using $n$-step mixed integer rounding functions

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ABSTRACT

The $n$-step mixed integer rounding (MIR) functions generate $n$-step MIR inequalities for MIP problems and are facets for the infinite group problems. We show that the $n$-step MIR functions also directly generate facets for the finite master cyclic group polyhedra especially in many cases where the breakpoints of the $n$-step MIR function are not necessarily at the elements of the group (hence the linear interpolation of the facet coefficients obtained has more than two slopes).



## 1. Introduction

Gomory (1969) introduced the group relaxation approach to integer programming (IP) and showed that facets of group polyhedra are sources for generating valid inequalities for IP problems. Many fundamental results in this regard were presented in Gomory (1969) and Gomory and Johnson (1972a,b). Development of facets for finite and infinite group polyhedra has been studied in many recent publications such as Aráoz et al. (2003), Dash and Günlük (2006a,b), Dey and Richard (2007), Gomory and Johnson (2003), Gomory et al. (2003), Kianfar and Fathi (2009), Klabjan (2007) and Richard et al. (2009).

Facets of the polyhedra defined by the cyclic group relaxation of a single IP constraint have been of particular interest; see Gomory (1969); Gomory and Johnson (1972a,b, 2003). If $C_N = \{0, \frac{1}{N}, \ldots, \frac{N-1}{N}\}$ denotes the cyclic group of order $N$ with addition modulo 1, the *finite master cyclic group polyhedron* over $C_N$ with the right-hand side $\frac{r}{N} \in C_N$, $\frac{r}{N} \neq 0$, is defined as $P(C_N, \frac{r}{N}) = conv\{(x_1, \ldots, x_{N-1}) \in \mathbb{Z}_+^{N-1} : \sum_{j=1}^{N-1} \frac{j}{N} x_j \equiv \frac{r}{N} \bmod 1\}$, where *conv* means convex hull.

Following the same concept, the *infinite group problem* is defined over the group of real numbers, $U = [0, 1)$, with addition modulo 1. For $u_0 \in U$, $u_0 \neq 0$, let $X(U, u_0)$ be the set of integer-valued functions $x(u)$ on $U$ with finite support such that $\sum_{u \in U} ux(u) \equiv u_0 \bmod 1$. Then the infinite group polyhedron with the right-hand side $u_0$ is defined as $P(U, u_0) = conv\{x(u) \in X(U, u_0)\}$.

In Kianfar and Fathi (2009), we introduced the $n$-step mixed integer rounding (*MIR*) *functions* and proved that they are facet-defining for the infinite group problem. In this paper we further those results and show that these functions can also *directly* generate facets for the *finite* master cyclic group polyhedra. In other words, we show that we get facets simply by taking the values of the $n$-step MIR function (with appropriate parameters) at the elements of $C_N$. We will note that in the case where the breakpoints of the $n$-step MIR function are all at the elements of $C_N$ (the function obtained by linear interpolation of the facet coefficients is two-slope), this is a direct result of the properties proved in Gomory and Johnson (1972b) (see Section 3). However, we will show that the facet-defining property of these functions is also true for many cases where the breakpoints of the $n$-step MIR function are not necessarily at the elements of $C_N$ (and therefore the linear interpolation of the facet coefficients can have more than two slopes).

For the special case of 2-step MIR functions, this result was previously proved in Dash and Günlük (2006a), although the line of argument used there was different from the one that we employ here.

In Section 2 we review the $n$-step MIR functions and some relevant results from Kianfar and Fathi (2009). In Section 3 we show our main result, and in Section 4 we discuss the extension to the polyhedra with continuous variables.

## 2. Background

In Kianfar and Fathi (2009), we presented the *$n$-step MIR inequalities* for the feasible set of a general IP constraint, i.e.

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$$Y = \left\{ (x_1, \ldots, x_{|J|}) \in \mathbb{Z}_+^{|J|} : \sum_{j \in J} a_j x_j + z = b,\ z \in \mathbb{Z} \right\},$$

where $b \notin \mathbb{Z}$ and $J$ is the index set. They showed that each $n$-step MIR inequality is easily generated by applying a (positive or negative) *n-step MIR function* on $a_j$'s. Here we bring the definition of the $n$-step MIR function from Kianfar and Fathi (2009) with a slight change in notation. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$, where $\alpha_i > 0$ for all $i$. For $\beta \in \mathbb{R}$, let $\beta^{(\alpha_0)} = \beta$. Then, for $n \geqslant 1$ we define $\beta^{(\alpha_n)}$ recursively as

$$\beta^{(\alpha_n)} = \beta^{(\alpha_{n-1})} - \alpha_n \lfloor \beta^{(\alpha_{n-1})} / \alpha_n \rfloor.$$

Now define the subsets $I_0, \ldots, I_n \subset \mathbb{R}$ for any given $n$, $b$ and $\alpha$ as follows:

- $I_m = \{u \in \mathbb{R} : u^{(\alpha_i)} < b^{(\alpha_i)},\ i = 1, \ldots, m,\ u^{(\alpha_{m+1})} \geqslant b^{(\alpha_{m+1})}\}$, for $m = 0, \ldots, n-1$;
- $I_n = \{u \in \mathbb{R} : u^{(\alpha_i)} < b^{(\alpha_i)},\ i = 1, \ldots, n\}$.

**Definition 1.** Let $n \in \mathbb{N},\ b \in \mathbb{R}$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$, where $\alpha_i > 0$ for all $i$. If $b/\alpha_1 < \lceil b/\alpha_1 \rceil$ and $b^{(\alpha_{i-1})}/\alpha_i < \lceil b^{(\alpha_{i-1})}/\alpha_i \rceil \leqslant \alpha_{i-1}/\alpha_i$ for $i = 2, \ldots, n$, then the *positive n-step MIR function* for the right-hand side $b$ is defined as

$$g_+^{\alpha,b}(u) = \frac{\alpha_1 \delta^{n,b}(u) - u^{(\alpha_1)} \prod_{l=2}^{n} \left\lceil \frac{b^{(\alpha_{l-1})}}{\alpha_l} \right\rceil}{(\alpha_1 - b^{(\alpha_1)}) \prod_{l=2}^{n} \left\lceil \frac{b^{(\alpha_{l-1})}}{\alpha_l} \right\rceil}, \tag{1}$$

where $\delta^{n,b}(u)$ is

- $\prod_{l=2}^{n} \left\lceil \frac{b^{(\alpha_{l-1})}}{\alpha_l} \right\rceil$ if $u \in I_0$;
- $\sum_{i=2}^{m} \left( \prod_{l=i+1}^{n} \left\lceil \frac{b^{(\alpha_{l-1})}}{\alpha_l} \right\rceil \right) \left\lfloor \frac{u^{(\alpha_{i-1})}}{\alpha_i} \right\rfloor + \left( \prod_{l=m+2}^{n} \left\lceil \frac{b^{(\alpha_{l-1})}}{\alpha_l} \right\rceil \right) \left\lceil \frac{u^{(\alpha_m)}}{\alpha_{m+1}} \right\rceil$ if $u \in I_m$, $m = 1, \ldots, n-1$;
- $\sum_{i=2}^{n} \left( \prod_{l=i+1}^{n} \left\lceil \frac{b^{(\alpha_{l-1})}}{\alpha_l} \right\rceil \right) \left\lfloor \frac{u^{(\alpha_{i-1})}}{\alpha_i} \right\rfloor + \frac{u^{(\alpha_n)}}{b^{(\alpha_n)}}$ if $u \in I_n$.

Also if $-b/\alpha_1 < \lceil -b/\alpha_1 \rceil$ and $(-b)^{(\alpha_{i-1})}/\alpha_i < \lceil (-b)^{(\alpha_{i-1})}/\alpha_i \rceil \leqslant \alpha_{i-1}/\alpha_i$ for $i = 2, \ldots, n$, then the *negative n-step MIR function* for the right-hand side $b$ is defined as

$$g_-^{\alpha,b}(u) = g_+^{\alpha,-b}(-u). \tag{2}$$

It is easy to verify that $g_+^{\alpha,b}(u)$ is a two-slope piecewise linear continuous function. It is also periodic in $u$ and $b$ with period $\alpha_1$. In other words $g_+^{\alpha,b+k_b\alpha_1}(u + k_u\alpha_1) = g_+^{\alpha,b}(u)$ for $k_b, k_u \in \mathbb{Z}$. Of course $g_-^{\alpha,b}(u)$ also has all these properties.

The formal statement of the result regarding $n$-step MIR inequalities is as follows (Corollary 1 in Kianfar and Fathi (2009)). If $\alpha_1 = 1/t,\ t \in \mathbb{N}$, and $n$, $b$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ satisfy the conditions required for the definition of $g_+^{\alpha,b}(\cdot)$, then the *positive n-step MIR inequality* $\sum_{j \in J} g_+^{\alpha,b}(a_j) x_j \geqslant 1$ is valid for the set $Y$. Also, if these parameters satisfy the conditions required for the definition of $g_-^{\alpha,b}(\cdot)$, then the *negative n-step MIR inequality* $\sum_{j \in J} g_-^{\alpha,b}(a_j) x_j \geqslant 1$ is valid for $Y$.

The result from Kianfar and Fathi (2009) of interest in this paper is that the $n$-step MIR inequalities when generated for $P(U, u_0)$ are not only valid but also facet-defining. A non-trivial *valid inequality* for $P(U, u_0)$ is completely defined by its coefficients, and hence can be defined as a real-valued function $\pi$ defined for all $u \in U$ such that $\pi(0) = 0,\ \pi(u) \geqslant 0,\ u \in U$ and $\sum_{u \in U} \pi(u) x(u) \geqslant 1$ for any $x(u) \in X(U, u_0)$. A valid inequality $\pi$ for $P(U, u_0)$ is a *facet* (*extreme* valid inequality) if $\pi$ cannot be written as a strict convex combination of two distinct valid inequalities for $P(U, u_0)$. We have the following result from Theorem 9 in Kianfar and Fathi (2009).

**Theorem 1.** Kianfar and Fathi (2009). *Let $u_0 \in U,\ n, t \in \mathbb{N}$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that $\alpha_1 = 1/t,\ \alpha_i \in \mathbb{R},\ \alpha_i > 0$ for $i = 2, \ldots, n$. If $u_0/\alpha_1 < \lceil u_0/\alpha_1 \rceil$ and $u_0^{(\alpha_{i-1})}/\alpha_i < \lceil u_0^{(\alpha_{i-1})}/\alpha_i \rceil \leqslant \alpha_{i-1}/\alpha_i$ for $i = 2, \ldots, n$, then the function $g_+^{\alpha,u_0}(u)$ defines a facet for $P(U, u_0)$. Similarly, if $-u_0/\alpha_1 < \lceil -u_0/\alpha_1 \rceil$ and $(-u_0)^{(\alpha_{i-1})}/\alpha_i < \lceil (-u_0)^{(\alpha_{i-1})}/\alpha_i \rceil \leqslant \alpha_{i-1}/\alpha_i$ for $i = 2, \ldots, n$, the function $g_-^{\alpha,u_0}(u)$ defines a facet for $P(U, u_0)$.*

## 3. Using *n*-step MIR functions to generate facets for finite master cyclic group polyhedra

In this section we show that $n$-step MIR functions directly generate facets for the finite master cyclic group polyhedra especially in many cases where the breakpoints of the $n$-step MIR function are not necessarily elements of $C_N$. Note that the set used in the definition of $P(C_N, \frac{r}{N})$ is equivalent to a special case of the set $Y$, where $J = \{1, \ldots, N-1\},\ a_j = \frac{j}{N}$, and $b = \frac{r}{N}$. We denote this set by $Y_C$. Therefore $Y_C = \left\{ (x_1, \ldots, x_{N-1}) \in \mathbb{Z}_+^{N-1} : \sum_{j=1}^{N-1} \frac{j}{N} x_j + z = \frac{r}{N},\ z \in \mathbb{Z} \right\}$, and $P(C_N, \frac{r}{N}) = conv(Y_C)$. The vector $\pi = (\pi_1, \ldots, \pi_{N-1}) \in \mathbb{R}_+^{N-1}$ defines a valid inequality for $P(C_N, \frac{r}{N})$ if $\sum_{j=1}^{N-1} \pi_j x_j \geqslant 1$ for every $(x_1, \ldots, x_{N-1}) \in Y_C$. A valid inequality $\pi$ for $P(C_N, \frac{r}{N})$ defines a facet if $\pi$ cannot be written as a strict convex combination of two distinct valid inequalities for $P(C_N, \frac{r}{N})$.

The following result from Gomory and Johnson (1972b) states a relationship between facets of the infinite group polyhedra and finite master cyclic group polyhedra.

**Theorem 2.** Gomory and Johnson (1972b). *If $\rho(u)$ defines a facet for $P(U, u_0)$, where $u_0 = \frac{r}{N}$, and it consists of straight line segments connected at values $u = \frac{j}{N}$ for $j = 0, \ldots, N$, then $\pi = (\pi_1, \ldots, \pi_{N-1})$, where $\pi_j = \rho\left(\frac{j}{N}\right)$, defines a facet for $P(C_N, \frac{r}{N})$.*

In the case that the breakpoints of the piecewise linear $n$-step MIR function fall onto the elements of $C_N$, the facet-generating property of this function for $P(C_N, \frac{r}{N})$ is a direct result of Theorem 2. We formally state this property in Lemma 1 since we will use it in proving our main result in Theorem 4.

**Lemma 1.** *Let $\alpha = (\alpha_1, \ldots, \alpha_n) = \left(\frac{d_1}{N}, \frac{d_2}{N}, \ldots, \frac{d_n}{N}\right)$ be the parameter vector, where $n, N, d_i \in \mathbb{N},\ i = 2, \ldots, n$ and $\frac{N}{d_1} \in \mathbb{N}$. Also let $b = \frac{r}{N}$ be the right-hand side, where $r \in \mathbb{N}$ and $r < N$. If $b/\alpha_1 < \lceil b/\alpha_1 \rceil$ and $b^{(\alpha_{i-1})}/\alpha_i < \lceil b^{(\alpha_{i-1})}/\alpha_i \rceil \leqslant \alpha_{i-1}/\alpha_i$ for $i = 2, \ldots, n$, then $\pi = (\pi_1, \ldots, \pi_{N-1})$, where $\pi_j = g_+^{\alpha,\frac{r}{N}}\left(\frac{j}{N}\right)$, defines a facet for $P(C_N, \frac{r}{N})$. Similarly, if $-b/\alpha_1 < \lceil -b/\alpha_1 \rceil$ and $(-b)^{(\alpha_{i-1})}/\alpha_i < \lceil (-b)^{(\alpha_{i-1})}/\alpha_i \rceil \leqslant \alpha_{i-1}/\alpha_i$ for $i = 2, \ldots, n$, then $\pi = (\pi_1, \ldots, \pi_{N-1})$, where $\pi_j = g_-^{\alpha,\frac{r}{N}}\left(\frac{j}{N}\right)$, defines a facet for $P(C_N, \frac{r}{N})$.*

It is easy to see why Lemma 1 is true: Let $\pi = (\pi_1, \ldots, \pi_{N-1})$, where $\pi_j = g_+^{\alpha,\frac{r}{N}}\left(\frac{j}{N}\right)$. By Theorem 1, the function $g_+^{\alpha,\frac{r}{N}}(u)$ defines a facet for $P(U, \frac{r}{N})$. Moreover, this function is a piecewise linear continuous function. The breakpoints of this function happen at the boundary points of the sets $I_m$, $m = 0, 1, \ldots, n$. These are the points at which either $u^{(\alpha_i)} = 0$ or $u^{(\alpha_i)} = \left(\frac{r}{N}\right)^{(\alpha_i)}$. Now since 0, $\frac{r}{N}$ and $\alpha_i$, for $i = 1, \ldots, n$, are all integer multiples of $\frac{1}{N}$, all these boundary points and hence the breakpoints of the function $g_+^{\alpha,\frac{r}{N}}(u)$ occur on the elements of the group $C_N$. Therefore, $g_+^{\alpha,\frac{r}{N}}(u)$ over $U$ consists of straight line segments connected at values $u = \frac{j}{N}$ for $j = 0, \ldots, N$. Thus, by Theorem 2, $\pi$ defines a facet for $P(C_N, \frac{r}{N})$. A similar argument proves the result for the negative $n$-step MIR function.

**Example 1.** Let $\alpha = \left(\frac{20}{20}, \frac{5}{20}, \frac{2}{20}\right)$ and $b = \frac{8}{20}$. These values of $\alpha$ and $b$ satisfy the conditions of Lemma 1. Therefore according to this lemma, $\pi = (\pi_1, \ldots, \pi_{19})$, where $\pi_j = g_+^{\alpha,b}\left(\frac{j}{20}\right)$ defines a facet for $P(C_{20}, \frac{8}{20})$. In Fig. 1 the $\pi_j$ values of this facet for $j = 1, \ldots, 19$ are