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A provisioning problem with stochastic payments

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ABSTRACT

We consider the problem of determining the minimal requirement one must establish in order to meet a series of future random payments. It is shown in a very general setting that this problem can be recast as a chance constrained model and how the technique of Sample Average Approximation can be employed to find solutions. We also use comonotonic theory to analyze analytical approximations in a restricted Gaussian setting. Our numerical illustrations demonstrate that the Sample Average Approximation is a viable and efficient way to solve the stated problem generally and outperforms the analytical approximations. In passing we present a result that is related to Stein's famous lemma (Stein, 1981) and is of interest in itself.

1. Introduction and motivation

In this paper we consider the problem of an economic agent who aims at determining a minimal requirement (provision) R_0 such that it becomes likely that he will be able to meet his future obligations which are expressed as a series of future stochastic payments $\Theta_1, \Theta_2, \ldots, \Theta_n$ that are due at times $i = 1, 2, \ldots, n$.

We assume a "hold-to-maturity" approach which means that R_0 is determined such that upon investing it one will be able to pay-off in future, with a sufficiently high probability, the amounts Θ_i when they are due. For example in case of an insurance portfolio of life annuities, the total initial provision needs to be sufficient to ensure that rents can be paid to the annuitants as long as they survive.

Such a "hold-to-maturity" (actuarial) approach to assess solvency contrasts with a "what-if-I-sell" (financial) approach, where a so-called market value L_0 for the series of obligations will be determined first, which is akin to calculating the amount one needs to put aside when selling these obligations to a third party. Hence in such setting L_0 is determined with reference to financial markets and this will typically involve non-arbitrage techniques and so-called risk neutral probabilities (Harrison and Kreps, 1979). Next the minimal requirement R_0 is established such that for a given risk horizon (typically 1 year) a potential loss in the market value can be absorbed with a sufficiently large probability. Going back to our example in life insurance, the agent determines the value L_0 first and next he obtains the total provision R_0 by adding to L_0 a buffer C_0 such that next year, with sufficiently high probability, he is still going to be able to transfer the obligations to a third party under adverse events.

While the latter financial approach underpins the latest modern regulatory risk management frameworks such as Basel II and Solvency II it is subject to the criticism that economic agents are not necessarily assuming a "what-if-I-sell" attitude when assessing the risks, but may rather operate under a going concern situation. We use an example to further discuss this point: a risk manager of a life annuity insurance portfolio may want to back his future obligations with a carefully constructed hold-to-maturity portfolio of bonds, because in this way he can be reassured that he will be able to pay-off all the rents when they are due. Of course some of the (corporate) bonds may be subject to a default but the risk manager's strategy is that these possible losses will be compensated by the interest income (coupons) that he is earning over time. However, in turbulent market conditions there may be less demand than supply in the bond market (less liquidity) giving rise to declining bond prices, which has a negative impact on the market value of the portfolio and thus solvency position from a purely "what-if-I-sell" point of view. For the "hold-to-maturity" risk manager however changing liquidity does not necessarily implies a real threat because he is going to hold the portfolio until maturity anyway, and thus only cares about actual defaults. To put it sharply: why does the life insurance risk manager needs to be concerned about the market value of his portfolio next year if he is going to hold the contracts until maturity anyway, and his real concern is his capacity to pay the obligations when they are due?

All this does not mean that we believe that the market based approach is "wrong" as such. In fact it cannot be denied that it has greatly helped in identifying the solvency problems we have been witnessing in the financial sector in the period 2008–2009,

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but complementing such an approach with a "hold-to-maturity" approach appears useful and might better aligned with decision makers' behaviour. Hence our approach uses the real (physical) probabilities of the stochastic process under consideration, and we set up provisions such that the corresponding investment strategy avoids *inability to meet future obligations*, and guarantees that the available funds in every period is always above a given threshold (*the hurdles*) with some desired (high) probability.

The main contributions of the paper are then as follows: *Firstly*, we formulate a general stochastic provision problem and show it can be recast as a chance constrained problem. Next we demonstrate how Sample Average Approximation (SAA), which is a technique from the field of stochastic programming, can be applied to solve the resulting optimization problem. We discuss some theoretical properties of this approach and demonstrate its efficiency numerically. *Secondly*, we also explore analytical approximations in a more restricted Gaussian setting and show how approximate solutions can be derived. The approximations we use are based on the idea of taking conditional expectations, first proposed in finance by Rogers and Shi (1995). A series of papers, discussing a wide range of problems in finance and insurance, have shown that the resulting so-called comonotonic approximations allow to provide efficient and often accurate solutions to the problem at hand; see for instance Dhaene et al. (2008). However, technically speaking the literature so far has been considering sums of lognormals, leaving it as an open question whether the comonotonic approximations could also be useful to analyse other than lognormal sums. This paper seems to be first to show that comonotonic approximations can be applied to more complex random sums. *Thirdly*, when comparing SAA with the analytical approximations we find that the SAA performance is superior. Indeed, SAA is essentially a distribution-free technique which can be used in a fairly general multivariate setting whereas the comonotonic approximations are restricted to a multivariate normal setting. But even in such constrained setting SAA appears to provide more accurate results. *Finally*, in passing we present a result that allows easy calculation of the first moment for a certain product of random variables. This result has connection with Stein's celebrated lemma (Stein, 1981) and is of some interest in itself.

The rest of the paper is structured as follows. In Section 2 we describe the setting of our stochastic provisioning problem in detail. Next, in Section 3 we demonstrate that it can be formulated as a general chance constrained problem, and using ideas contained in Pagnoncelli et al. (2009) we show how SAA can be used to obtain solutions and bounds for determining the minimal provision. Further in Section 4 we generalize the results from Rogers and Shi (1995) as well as Kaas et al. (2000) from sums of lognormals to more general sums of random variables and derive approximate analytical solutions for the problem at hand. In Section 5 we present several numerical experiments and we compare both approaches. We show that in contrast to comonotonic approximations the SAA is a viable and efficient way to solve our stochastic provisioning problem. Finally, Section 6 provides final remarks.

2. Problem setting

Throughout this paper all random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All expectations and other statistical quantities are tacitly assumed to exist.

In our problem setting there is an initial provision R_0 which, once it has been established, is invested such that it generates in future periods $[i-1, i]$ a stochastic (log)return Y_i $(i = 1, 2, \ldots, n)$. Hence the at time j available amount $R_j(R_0)$, after payment of the amount Θ_j, is defined as

$$R_j(R_0) = R_{j-1}\exp(Y_j) - \Theta_j, \quad j = 1, \ldots, n. \tag{1}$$

By solving the recursion (1) we find the following expression for the available provision $R_j(R_0)$ at time j:

$$R_j(R_0) = R_0 e^{Y_1+\cdots+Y_j} - \sum_{i=1}^{j}\Theta_i e^{Y_{i+1}+\cdots+Y_j}, \quad j = 1, \ldots, n. \tag{2}$$

The requirement to be able to pay the obligations Θ_j $(j = 1, 2, \ldots, n)$ amounts to imposing that $R_j(R_0) \geqslant 0$ must hold for $j = 1, 2, \ldots, n$. However, we may need to consider some more additional constraints. Indeed internal policy or external controlling authorities may require the economic agent to build-up some buffers to back these obligations. Let the buffer that must be available at time j $(j = 0, 1, \ldots, n)$ be given by $V_j \geqslant 0$. The amounts V_j, which are not necessarily deterministic, can be interpreted as additional *hurdles* one must pass in order to stay 'in business', i.e. $R_j(R_0) \geqslant V_j$ should always hold $(j = 0, 1, \ldots, n)$. The natural idea is then to determine R_0 such that it becomes 'as small as possible, but still large enough to ensure that all hurdles V_j can be met'. In the remainder of the paper we use the notation $\overline{R}_0$ to denote such an optimal R_0. Hence the *joint hurdle-race* problem amounts to determining the initial provision $\overline{R}_0$ as the smallest amount R_0 that enables us to 'pass all hurdles simultaneously', with a significance level of $1-\varepsilon$, where $0 < \varepsilon < 1$ is chosen sufficiently small:

$$\begin{aligned} \overline{R}_0 &= \operatorname*{Min}_{R_0\in\mathbb{R}} R_0 \\ \text{s.t. } &\Pr\{R_j(R_0) \geqslant V_j, \quad j = 0, 1, \ldots, n\} \geqslant 1-\varepsilon. \end{aligned} \tag{3}$$

The multivariate constraint that appears in (3) reflects uniform safety requirements desired by a decision maker. It guarantees that for the entire period under consideration the available funds $R_j(R_0)$ will remain sufficiently high when monitored at intermediate times $j = 1, \ldots, n$.

Note that this joint-hurdle problem (3) is related with – but different from – a hurdle race problem that was formulated in Vanduffel et al. (2003). Indeed, these authors considered a *hurdle-race problem*as follows:

$$\begin{aligned} \overline{R}_0 &= \operatorname*{Min}_{R_0\in\mathbb{R}} R_0 \\ \text{s.t. } &\Pr\{R_j(R_0) \geqslant V_j\} \geqslant 1-\varepsilon_j, \quad j = 1, \ldots, n \end{aligned} \tag{4}$$

for given significance levels $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in [0,1]$. Hence they aim at determining the initial provision $\overline{R}_0$ as the smallest amount R_0 that enables to 'pass all hurdles separately', with predetermined significance levels of $1-\varepsilon_j$, where the ε_j are chosen sufficiently small (they typically have an order of magnitude equal to 5% or less).

There are three crucial differences between their formulation and ours. *Firstly*, the joint formulation (3) involves a single multivariate constraint whereas the original formulation involves n univariate constraints separately. We believe the joint formulation (3) reflects the proper safety requirements desired by the decision maker in a natural way. It guarantees that the available funds remain positive with high probability for the whole time period considered. In contrast formulation (4) assures that for each period the probability of remaining solvent is high, but the probability of having a shortfall *at least once* during the entire period under consideration may remain high. We believe such formulation might not reflect agents' objectives. Note however that while the joint formulation seems to be more appropriate this comes at the cost of significant mathematical complexity. Indeed while we have only a single constraint in (3) as opposed to n different ones in formulation (4), problem (3) is harder to solve because dealing with the vector $(R_1(R_0), \ldots, R_n(R_0))$ is a difficult task since it involves sums of dependent random variables taken jointly. *Secondly*, the original hurdle-race problem was limited to fixed payments Θ_i $(i = 1, 2, \ldots, n)$ in a Gaussian setting (see Vanduffel et al., 2003) meaning that available comonotonic theory could be readily