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# Valuing finite-lived Russian options ☆

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**Abstract**

This paper deals with the valuation of the Russian option with finite time horizon in the framework of the Black–Scholes–Merton model. On the basis of the PDE approach to a parabolic free boundary problem, we derive Laplace transforms of the option value, the early exercise boundary and some hedging parameters. Using Abelian theorems of Laplace transforms, we characterize the early exercise boundary at a time to close to expiration as well as the well-known perpetual case in a unified way. Furthermore, we obtain a symmetric relation in the perpetual early exercise boundary. Combining the Gaver–Stehfest inversion method and the Newton method, we develop a fast algorithm for computing both the option value and the early exercise boundary in the finite time horizon.


*Keywords:* Finance; Russian options; Optimal stopping problem; Free boundary problem; Laplace transforms

## 1. Introduction

Russian options are *path-dependent* contingent claims that give the holder the right to receive the realized supremum value of the underlying asset prior to his exercise time. The holder can exercise the option at any time, *i.e.*, the option is of American-style. Shepp and Shiryaev (1993, 1994) introduced the Russian option, assuming no maturity date for the exercise; see Duffie and Harrison (1993) for a financial justification of their results. Shepp and Shiryaev showed that there exists an optimal threshold level of the asset price below which it is advantageous to exercise the option, provided that the asset pays dividends. This type of Russian options can be regarded as a special case of American lookback options. More specifically, it is the perpetual fixed-strike lookback call option with null strike price (Pedersen, 2000). In common with lookback options, Russian options are *not* genuine option contracts, because they pay the holder the supremum asset price, always finishing in-the-money. This means that high premiums are charged for Russian options in compensation for "reduced regret" (Shepp and Shiryaev, 1993).

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This paper deals with Russian options with *finite time horizon*, *i.e.*, there is a finite expiry or maturity date for the exercise. The holder may exercise the option at any time but during the option's lifetime. Recently, some researchers have contributed theoretical results to the valuation of finite-lived Russian options. Ekström (2004) showed the existence and continuity of an *optimal stopping* or *early exercise* boundary for the Russian option. Also, Ekström proved that the option value is given by the solution of a certain boundary value problem, from which he analyzed asymptotic behavior of the optimal stopping boundary near expiration. This free boundary problem was further studied by Duistermaat et al. (2005) who suggested a numerical algorithm for valuing the Russian option; see Kyprianou and Pistorius (2003) for related theoretical work. Peskir (2005) proved that the optimal stopping boundary can be characterized by the solution of a nonlinear integral equation arising from the early exercise premium representation.

Except for Duistermaat et al. (2005), there is no quantitative research of the finite-lived Russian option, which is principally due to the lack of efficient tools for solving the free boundary problem. Duistermaat et al. have used the method of randomization of Carr (1998) who proposed that the value of an American vanilla option can be approximated by a randomization of the maturity date using an $n$-stage Erlangian distribution. As $n \to \infty$, it is possible to show convergence to the value of the American option. This idea has its origin in the classic theory of integral transforms, and it goes by the name of the Post-Widder inversion formula (Widder, 1946); see Abate and Whitt (1992, Section 8) for an algorithm based on the Fourier series. Duistermaat et al. (2005) developed a recursive algorithm for computing the $n$th approximations of the value and the early exercise boundary of the finite-lived Russian option. The complexity of their algorithm comes from the expression of the $n$-stage Erlangian distribution, and it is directly concerned with the implementation and speed of the algorithm. The purpose of this paper is to provide another quantitative method for computing both the option value and the early exercise boundary.

This paper is organized as follows: in Section 2, we formulate the valuation problem both as an optimal stopping problem and a free-boundary problem, obtaining a partial differential equation (PDE) and its boundary conditions. In Section 3, applying the Laplace transform approach to this PDE, we obtain explicit Laplace transforms of the option value and its Greeks, which include the transformed early exercise boundary characterized implicitly through a functional equation. Abelian theorems of Laplace transforms enable us to obtain an asymptotic behavior of the early exercise boundary near expiration as well as the well-known perpetual result in a unified way. Furthermore, motivated by some observations in numerical experiments, we obtain an interesting symmetric relation in the perpetual early exercise boundary. To see the power of our Laplace transform approach, we develop in Section 4 a fast algorithm for computing both the option value and the early exercise boundary in the finite time horizon, providing some computational results for particular cases. Finally, in Section 5, we give concluding remarks and directions of future research.

## 2. Basic framework

### 2.1. Optimal stopping problem

The setup is the standard Black–Scholes–Merton framework where the price of the underlying asset evolves according to a geometric Brownian motion: Let $(S_t)_{t\geqslant 0}$ be the price process of the underlying asset, which is defined by

$$S_t = s\exp\left\{\left(r-\delta-\frac{1}{2}\sigma^2\right)t+\sigma W_t\right\}, \quad t\geqslant 0, \tag{1}$$

where $S_0 = s > 0$, $r > 0$ is the risk-free rate of interest, $\delta \geqslant 0$ is the continuous dividend rate, $\sigma > 0$ is the volatility coefficient of the asset price, and $W \equiv (W_t)_{t\geqslant 0}$ is a one-dimensional standard Brownian motion process on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbb{P})$. The filtration $\mathbb{F} \equiv (\mathscr{F}_t)_{t\geqslant 0}$ is a natural one generated by $W$ and the probability measure $\mathbb{P}$ is chosen so that the stock has mean rate of return $r$. For the price process $(S_t)_{t\geqslant 0}$ and a constant $m \geqslant s$, define the supremum process as

$$M_t = m \vee \sup_{0\leqslant u\leqslant t} S_u, \quad t\geqslant 0, \tag{2}$$

where $a \vee b = \max\{a, b\}$.