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Continuous Optimization

A largest-distance pivot rule for the simplex algorithm ☆

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Abstract

We propose a new pivot rule for the simplex algorithm, which is demonstrative in the dual space intuitively. Although it is based on normalized reduced costs, like the steepest-edge rule and its variants, the rule is much simpler and cheaper than the latter. We report computational results obtained with the 47 largest Netlib problems in terms of the number of rows and columns, all of the 16 Kennington problems, and the 17 largest BPMPD problems. Over the total 80 problems, a variant of the rule outperformed the Devex rule with iterations and time ratio 1.43 and 3.24, respectively.

Keywords: Large-scale linear programming; Simplex algorithm; Pivot rule; Largest-distance; Scaling

1. Introduction

To simplify our exposition, we consider the linear programming (LP) problem in the standard form:

$$\begin{aligned} &\text{Minimize} \quad c^{\mathrm{T}}x, \\ &\text{subject to} \quad Ax = b, \quad x \geqslant 0, \end{aligned} \tag{1.1}$$

where $A \in R^{m\times n}$ $(m < n)$, $b \in R^m$, $c \in R^n$ and rank$(A) = m$. It will be a simple matter to extend results of this paper to more general LP problems with bounds and ranges.

The pivot rule used to select an entering index in the simplex algorithm is crucial to algorithm's performance, as it essentially determines the number of iterations required for solving LP problems.

Dantzig's original rule selects an entering index among all nonbasic ones that corresponds to the most negative reduced cost. The idea behind the rule is to result in the maximum improvement in the objective value per unit steps in the entering variable. Somewhat surprisingly, however, an iteration reduction is observed with partial pricing rule, which only selects an entering index among a small part of the nonbasic indices at each iteration. Harris wrote: "this observation gives yet further confirmation of the importance of not seeking out negative costs of large modulus." [5].

Various rules have been proposed and tested in the past to improve efficiency (see [6,1,9,10,7], among others) until the *steepest-edge* rule and its variants emerged, and won the victory [5,4,3]. The

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merit of the steepest-edge rule, which selects an entering index based on *normalized* reduced costs, lies in its selection of the most downhill from among the candidates, giving the largest consequent decrease in the objective value per unit distance moved along the edge. The steepest-edge rule requires more computation in each iteration, but this computation is more than offset by the iteration reduction, if the normalization factors are computed in a recurrence manner.

In this paper, we present another pivot rule based on normalized reduced costs. The rule is of interesting meanings in the dual space intuitively, and is also simple and cheap, relative to the steepest-edge rule and its variants. The rule uses *fixed* normalized factors in the solution process, and can even be implemented by simple scaling. We report encouraging computational results against Devex rule on 80 test problems, including the 47 largest Netlib problems in terms of the number of rows and columns, all of the 16 Kennington problems, and the 17 largest BPMPD problems in terms of more than 500 KB in compressed form.

To make the paper self-contained, we first review Dantzig's original rule, the steepest-edge rule and Harris' Devex rule briefly in the following three sections. Then we describe the largest-distance rule in Section 5. Finally, we report extensive computational results in Section 6.

2. Dantzig's original rule

Let B be the current basis and N the associated nonbasis. Without confusion, denote *basic* and *nonbasic* index sets again by B and N, respectively. The reduced costs associated with nonbasic indices may be computed by:

$$\bar{c}_N = c_N - N^{\mathrm{T}}\pi, \tag{2.1}$$

where π is the simplex multipliers defined by:

$$B^{\mathrm{T}}\pi = c_B. \tag{2.2}$$

Introduce index set:

$$J = \{j | \bar{c}_j < 0,\ j \in N\}. \tag{2.3}$$

If J is nonempty, Dantzig's original rule [2] selects an entering index q such that:

$$\bar{c}_q = \min\{\bar{c}_j | j \in J\} < 0. \tag{2.4}$$

3. Steepest-edge rule

For simplicity, assume that basis B consists of the first m columns of the constraint matrix. Then the set of edge directions can be written as:

$$d_j = \begin{bmatrix} -B^{-1}N \\ I \end{bmatrix} e_{j-m}, \quad j \in N, \tag{3.1}$$

where $I \in R^{(n-m)\times(n-m)}$ and e_i is the unit $(n-m)$-vector with its ith component one. Note that $\bar{c}_j = c^{\mathrm{T}}d_j$. When J is nonempty, the steepest-edge rule chooses an entering index q such that:

$$\bar{c}_q/\|d_q\|_2 = \min\{\bar{c}_j/\|d_j\|_2 | j \in N\} < 0. \tag{3.2}$$

Therefore, edge direction d_q is most downhill among edges with respect to the objective. If implemented explicitly, computation of the edge direction norms is too expensive to be practicable. Goldfarb and Reid [4] derived recurrence formulas for direction norms. If q replaces p in the basis, then recurrence formulas of the edge directions are easily derived:

$$\bar{d}_p = -(1/\alpha_q)d_q, \tag{3.3}$$

$$\bar{d}_j = d_j - (\alpha_j/\alpha_q)d_q, \quad j \in N,\ j \neq q. \tag{3.4}$$

Denote the jth column of A by a_j. It is obtained from (3.3) and (3.4) that:

$$\|\bar{d}_p\|_2^2 = \left(1/\alpha_q^2\right)\|d_q\|_2^2, \tag{3.5}$$

$$\|\bar{d}_j\|_2^2 = \|d_j\|_2^2 - 2(\alpha_j/\alpha_q)a_j^{\mathrm{T}}v + (\alpha_j/\alpha_q)^2\|d_q\|_2^2, \tag{3.6}$$

$$j \in N,\ j \neq q, \tag{3.7}$$

where

$$B^{\mathrm{T}}\pi_p = e_p, \quad \alpha_q = \pi^{\mathrm{T}}a_q, \quad \alpha_j = \pi_p^{\mathrm{T}}a_j, \quad Bw = a_q,$$

$$B^{\mathrm{T}}v = w. \tag{3.8}$$

4. Devex rule

Harris' Devex Code [5] uses an approximate steepest-edge rule, in which the norms $\|d_j\|_2$ of the edge directions are replaced by approximate weights w_j. Initially and periodically, a so called "reference framework" is set to the current set of nonbasic indices, and all weights w_j are set to one for all j in the set. At the other iterations, it uses weights w_j to approximate the norms of the subvectors $\hat{d}_j$ consisting of only those components of