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Recovering risk-neutral probability density functions from options prices using cubic splines and ensuring nonnegativity

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Abstract

We present a new approach to estimate the risk-neutral probability density function (pdf) of the future prices of an underlying asset from the prices of options written on the asset. The estimation is carried out in the space of cubic spline functions, yielding appropriate smoothness. The resulting optimization problem, used to invert the data and determine the corresponding density function, is a convex quadratic or semidefinite programming problem, depending on the formulation. Both of these problems can be efficiently solved by numerical optimization software.

In the quadratic programming formulation the positivity of the risk-neutral pdf is heuristically handled by posing linear inequality constraints at the spline nodes. In the other approach, this property of the risk-neutral pdf is rigorously ensured by using a semidefinite programming characterization of nonnegativity for polynomial functions.

We tested our approach using data simulated from Black–Scholes option prices and using market data for options on the S&P 500 Index. The numerical results we present show the effectiveness of our methodology for estimating the risk-neutral probability density function.

Keywords: Option pricing; Risk-neutral density estimation; Cubic splines; Quadratic programming; Semidefinite programming

1. Introduction

The risk-neutral probability measure is a fundamental concept in arbitrage pricing theory. By definition, a risk-neutral probability measure (RNPM) is a measure under which the current price of each security in the

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economy is equal to the present value of the discounted expected value of its future payoffs given a risk-free interest rate. Fundamental theorems of asset pricing indicate that RNPMs are guaranteed to exist under an assumption of no arbitrage.

If a unique RNPM on the space of future states of an economy is given, we can price any security for which we can determine the future payoffs for each state in the state space. Therefore, a fundamental problem in asset pricing is the identification of a risk-neutral probability measure. While the dynamics of an economy and the parameters for its stochastic models are not directly observable, one can infer some information about these dynamics from the current prices of the securities in this economy. In particular, one can extract one or more implied risk-neutral densities of the future price of a security that are consistent with the prices of options written on that security. When there are multiple RNPMs consistent with the observed prices, one may try to choose the "best" one, according to some criterion. We address this problem in this article using optimization models.

For a stock or index, the set of possible future states can be represented as an interval or ray, discretized if appropriate or necessary. In most cases, the number of states in this state space is much larger than the number of observed prices, resulting in a problem with many more variables than equations. This underdetermined problem has many potential solutions and we cannot obtain an unique or sensible solution without imposing some additional structure into the risk-neutral probability measure we are looking for.

The type of additional structure imposed has been the differentiating feature of the existing approaches to the problem of identifying implied RNPMs. These approaches can be broadly classified as parametric and nonparametric techniques and are reviewed by Jackwerth [18], and Bondarenko [9] (see also Section 2). Parametric methods choose a distribution family (or a mixture of distributions) and then try to identify the parameters for these distributions that are consistent with the observed prices [6,22]. In nonparametric techniques, one achieves more flexibility by allowing general functional forms. Structure is introduced either using prior distributions or smoothness restrictions. Our approach fits into this last category and we ensure the desired smoothness of the RNPM using spline functions.

Spline functions are piecewise polynomial functions that assume a predetermined value at certain points (*knots*) and satisfy certain smoothness properties. Other authors have also used spline fitting techniques in the context of risk-neutral density estimation, see [3,13]. In contrast to existing techniques, we allow the displacement of spline knots in a superset of the set of points corresponding to option strikes. The additional set of knots makes our model flexible and we use this flexibility to optimize the fit of the spline function to the observed prices. The basic formulation, without requiring the nonnegativity of the risk-neutral probability density function (pdf), is a convex quadratic programming (QP) problem.

Two strategies to impose the nonnegativity of the RNPM are presented and discussed in this paper. The first and simpler strategy is to require the estimated pdf to remain nonnegative at the spline nodes. This scheme maintains the QP structure of the problem since it brings only linear inequality constraints to the basic formulation. However, there is no guarantee of nonnegativity between the spline nodes. Our second approach replaces the basic QP formulation with a semidefinite programming (SDP) formulation but rigorously ensures the nonnegativity of the estimated pdf in its entire domain. It is based on an SDP characterization of nonnegative polynomial functions due to Bertsimas and Popescu [5] and requires additional variables and linear equality constraints as well as semidefiniteness constraints on some matrix variables. To our knowledge, this is the first spline function approach to risk-neutral density estimation with a positivity guarantee.

The rest of this paper is organized as follows: In Section 2, we provide the definition of RNPMs and briefly discuss some of the existing approaches. In Section 3, we discuss our spline approximation approach to RNPMs and develop our basic QP optimization model. The treatment of nonnegativity is given in Section 4. Section 5 is devoted to a numerical study of our approach both with simulated and market data. We provide a brief conclusion in Section 6.

2. Risk-neutral probability measures

We consider the following one-period economy: There are n securities whose current prices are given by s_0^i for $i = 1, \ldots, n$. At the end of the current period, the economy will be in one of the states from the state space Ω. If the economy reaches state $\omega \in \Omega$ at the end of the current period, security i will have the payoff $s_1^i(\omega)$. We