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Stochastics and Statistics

Weighted search games

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Abstract

In this paper we shall deal with search games in which the strategic situation is developed on a lattice. The main characteristic of these games is that the points in each column of the lattice have a specific associated weight which directly affects the payoff function. Thus, the points in different columns represent points of different strategic value. We solve three different types of games. The first involves search, ambush and mixed situations, the second is a search and inspection game and the last is related to the accumulative games.

Keywords: Game theory; Search theory; Two-person zero-sum game; Inspection game; Group decisions and negotiations

1. Introduction

Work on search theory began in the US Navy's Antisubmarine Warfare Operations Research Group in 1942 in response to the German submarine threat in the Atlantic. After several decades of development, search problems are still largely of the same form as in 1942: a single target is lost, and the problem is to find it efficiently with fixed resources.

Modelling the search-evasion contest as a two-person zero-sum game is suitable in circumstances where the target is aware of the search and seeks to avoid capture.

A two-person zero-sum game will be expressed by $G = (X, Y, M)$ where X and Y are the sets of pure strategies of Players I and II respectively, and

$$M : X \times Y \to \mathbb{R} \tag{1}$$

is the payoff function which represents the winnings of Player I and the losses of Player II. Player I chooses a strategy $A \in X$, Player II chooses a strategy $B \in Y$ and the A and B chosen determine the payoff $M(A,B)$ to Player I and $-M(A,B)$ to Player II.

A probability distribution on X, that is to say, a mixed strategy for Player I, will be written as a function $x : X \to \mathbb{R}$, such that $x(C) \geqslant 0$, for all $C \in X$, and $\sum_{C\in X} x(C) = 1$. Similarly, a mixed strategy for Player II will be given by a function $y : Y \to \mathbb{R}$, such that $y(C) \geqslant 0$, for all $C \in Y$, and $\sum_{C\in Y} y(C) = 1$. When the players use their mixed strategies x and y, the payoff $M(x,y)$ is the expected value of $M(A,B)$.

Tactical problems of search and ambush modelled as two-person zero-sum games have been widely studied in the literature [2,8,11]. These games are called search games (and when symmetric ambush games).

In this paper we shall deal with search games in which the strategic situation is developed in the lattice

$$L = \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}, \tag{2}$$

therefore the sets of strategies for the players, X and Y, are subsets of the set of subsets of L (the power set of L), $\mathscr{P}(L)$.

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Games in which the lattice L is the base set where the strategic situation is developed, are studied in [11,15]. The games studied in these papers are rectangular games for two players

$$G = (X, Y, M), \tag{3}$$

where $X \subset \mathscr{P}(L), Y \subset \mathscr{P}(L)$, and the payoff function is a function of the cardinalities of $A \in X, B \in Y$ and $A \cap B$. The results obtained by solving these games are applicable to situations other than that of a rectangle of two spacial dimensions, for instance $i = 1, 2, \dots, n$ can represent discrete time points, and $j = 1, 2, \dots, m$ an array of m ordered cells.

In all the games of this kind studied to date, all the points of L have the same importance, which is reflected in the fact that the payoff function is exclusively a function of the cardinalities of A, B and $A \cap B$. In this paper we will study rectangular games in the lattice L in which each of the columns $L_i = \{i\} \times \{1, 2, \dots, m\}, i = 1, 2, \dots, n$, has an assigned weight c_i, which is made clear by the fact that the payoff function is a function of the cardinalities of $A \cap L_i$, $B \cap L_i$ and $A \cap B \cap L_i$. Hence, points in different columns will represent points with different strategic value. In this way these games can model situations in which the search space where the strategic situation is developed can be considered as divided in zones with different strategic value, and the opponents can act in a different way in each zone. The most similar situation appearing in the literature corresponds to the infiltration and inspection games where the infiltrator has a safe zone. These games are studied in [1,3,6,8–10], in them there is just one point or zone of the search space which differs from the rest of the space because, the infiltrator cannot be detected by his opponent when he is set there.

Hereinafter the cardinality of a set C will be written as $|C|$.

Let $F \subset \mathscr{P}(L)$ represent the family of all the subsets of L with just one point in each column. Each $C \in F$ can be interpreted as a pathway from the first column of L to the last column which does not double back on itself and also as a walk on a linear set of m points at the moments $1, 2, \dots, n$. In all the games we are going to study here, one of the sets of strategies, X or Y, will be equal to F. This allows us to address their resolution using the method developed in [15] and applied successfully to solve different games in [14–16]. For the games dealt with in this paper, the above mentioned method is summarised in the following results.

On set L we can define the transformations

$$T_s : L \to L, s = 1, 2, \dots, n,$$
$$T_s((i,j)) = T_s(i,j) = \begin{cases} (i,j) & \text{if } i \neq s, \\ (s, j+1) & \text{if } i = s,\ j < m, \\ (s,1) & \text{if } i = s,\ j = m. \end{cases} \tag{4}$$

Transformation T_s translates the points in column s cyclically and maintains the rest of the points of L fixed.

We will represent also by T_s the induced transformation on $\mathscr{P}(L), T_s : \mathscr{P}(L) \to \mathscr{P}(L)$, and on $\mathscr{P}(\mathscr{P}(L))$.

By successive applications of transformation T_s we obtain $T_s^2 C = T_s(T_s(C)), T_s^3 C = T_s(T_s(T_s(C)))$ and in general $T_s^i C$ is the subset of L obtained from the set L by applying transformation T_s i consecutive times, $T_s^i C = T_s \cdot \overset{i}{\smile} \cdot T_s C$. Therefore the set $T_s^i C$ contains the same points that C except in column s, where it contains the points obtained from the points of C in column s when they are moved up i times, bearing in mind that this is a cyclic movement and if the point (s, m) is moved by T_s it is transformed in the point $(s, 1)$.

Now, given a subset $C \subset L$, let $\overline{C}$ be the class defined by

$$\overline{C} = \left\{D = T_1^{i_1} T_2^{i_2} \dots T_n^{i_n} C, \forall i_1, i_2, \dots, i_n \text{ integers}\right\},$$

The following theorems are proved in [15].

Theorem 1.1. *Let $G = (X, Y, M)$ be a game on the lattice L satisfying $X = F, Y \subset \mathscr{P}(L), T_s Y = Y, M(T_s A, T_s B) = M(A, B), (A \in X,\ B \in Y,\ s = 1, 2, \dots, n)$. An optimal strategy for Player I is the uniform distribution on $X = F$*

$$x(A) = x_F(A) = \frac{1}{|F|} = \frac{1}{m^n}, \quad A \in X.$$

Let $B_0 \in Y$ such that

$$\min_{B \in Y} M(x_F, B) = \min \frac{1}{m^n} \sum_{A \in F} M(A, B) = \frac{1}{m^n} \sum_{A \in F} M(A, B_0) = M(x_F, B_0). \tag{5}$$

Thus, an optimal strategy for Player II is the distribution on Y uniformly concentrated in $\overline{B}_0$:

$$y_{\overline{B}_0}(B) = \begin{cases} \frac{1}{|\overline{B}_0|} & \text{if } B \in \overline{B}_0 \\ 0 & \text{if } B \notin \overline{B}_0 \end{cases}$$

and the value of the game is $M(x_F, B_0)$ given by (5).