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A new exact method for the two-dimensional orthogonal packing problem ☆

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Abstract

The two-dimensional orthogonal packing problem (2*OPP*) consists in determining if a set of rectangles (items) can be packed into one rectangle of fixed size (bin). In this paper we propose two exact algorithms for solving this problem. The first algorithm is an improvement on a classical branch&bound method, whereas the second algorithm is based on a new relaxation of the problem. We also describe reduction procedures and lower bounds which can be used within enumerative methods. We report computational experiments for randomly generated benchmarks which demonstrate the efficiency of both methods: the second method is competitive compared to the best previous methods. It can be seen that our new relaxation allows an efficient detection of non-feasible instances.

Keywords: Two-dimensional orthogonal packing problem; Open-dimension packing problem; Cutting and packing; Exact method; Branch&bound

1. Introduction

The *two-dimensional orthogonal packing problem* (2*OPP*) consists in determining if a set of rectangles (*items*) can be packed into one rectangle of fixed size (*bin*). This problem occurs in industry when rectangular pieces of steel, wood, or paper have to be cut from a larger rectangle. It can also be used to model the layout of a newspaper. This problem belongs to the family of *cutting and packing* (C & P) problems, and more precisely to the category of *two-dimensional open-dimension problems (2ODP)* in the improved typology of C & P problems proposed by Wäscher et al. [26]. It is *NP-complete* [14], and is an issue in the *two-dimensional bin-packing* (2*BP*), *two-dimensional knapsack* (2*KP*), and *strip-packing* problems (2*SP*).

A 2*OPP* instance D is a pair (A, B). A is the set of items a_i to pack. $B = (W, H)$ is a bin of width W and height H. An item a_i has a width w_i and a height h_i ($w_i, h_i \in \mathbb{N}$). We consider the version of the problem in which the items cannot be rotated. The position of the item a_i, denoted by (x_i, y_i), corresponds to the coordinates of its bottom left-hand corner. We say that a 2*OPP* instance D is feasible if there is a solution for D.

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Two ways are used in literature for solving exactly this problem: a *classical* scheme (see [23]), which packs items one by one in the bin, and a method based on graph-theoretical concepts [11–13]. The latter was proposed by Fekete and Schepers and outperforms the classical methods, as it removes many redundancies occurring in the tree search. In this paper we propose a third way of tackling $2OPP$. For this purpose we introduce a new relaxation of the problem and propose two branch&bound methods. The first is an improvement on the classical scheme and the second is based on the new relaxation. Computational experiments show that the latter method is competitive compared to Fekete and Schepers.

The first algorithm proposed ($LMAO$) is an improvement on the method proposed by Martello and Vigo [23]. Instead of testing the packing of items in each possible coordinate, the algorithm enumerates the packings of items only in the left-most-downward position. It also tests the possibility of not packing any item in this position. Using $LMAO$, the same pattern cannot appear twice in the enumeration.

The second algorithm (*TSBP*) is based on a new relaxation of $2OPP$. The relaxation consists in cutting each item (w_i, h_i) into h_i strips of width w_i. Note that cutting items into strips has been used in previous papers to design algorithms for bin-packing or knapsack problems [1,4,22,25] but our relaxation is different. All strips pertaining to a given item have to be packed at the same x-coordinate, even if they are not contiguous. Thus, the resulting problem cannot be solved as a $1BP$. Using this relaxation, the problem consists in finding a suitable set of x-coordinates for the items.

In the first step of *TSBP* (*outer branch&bound method*), all solutions of the relaxed problem are enumerated. Each solution is a valid list of x-coordinates for the items. For each solution found, a second enumerative method is run (*inner branch&bound method*) to seek a solution for the initial problem (i.e., a set of y-coordinates). If there is no solution for the relaxed problem, the inner branch&bound method is not run.

Several methods for improving the results are described. First we show how the size of the original instance can be reduced using preprocessing methods. We also recall several results concerning lower bounds for $2BP$ and describe how they can be applied to $2OPP$. At each step of the enumeration, the quality of the lower bounds is improved. The idea is to make use of the packed items to transform the instance into another instance with larger items. Another way of improving our method is to handle redundancies effectively. In the outer branch&bound method, we avoid all symmetries related to the y-axis, but redundant solutions can still be reached. So we propose methods to detect equivalent patterns in the search tree. These methods are different from those proposed by Scheithauer [24], as they apply even when the y-coordinates are not fixed yet.

We report computational experiments testing our methods against randomly generated benchmarks, and compare our algorithms to the methods proposed by Martello and Vigo [23], and by Fekete and Schepers [13]. The computational results show that our methods are efficient, as TSBP outperforms Martello and Vigo [23] and is competitive compared to Fekete and Schepers [11,13]. It also transpires that for many non-feasible instances, the relaxed problem also has no solution. If the instance is feasible, the number of states enumerated is also dramatically reduced: many non-feasible configurations are not enumerated and the inner branch&bound method is rarely run more than once.

Section 2 is a short state of the art for $2OPP$. In Section 3 we show that several reduction procedures can be applied to the orthogonal packing problem, and also how they can be improved. Section 4 describes our lower bounds, whereas Section 5 is devoted to our new branch&bound method. Section 6 deals with redundancies in the search tree and how they can be avoided. In Section 7 we introduce the new benchmarks and we report computational experiments.

2. Literature review

Several lower bounds have been proposed for $2BP$, all of which can be used for $2OPP$. Martello and Vigo [23] and Boschetti and Mingozzi [5,6] propose lower bounds specially for this problem, whereas Fekete and Schepers [12] apply one-dimensional functions called *Dual-Feasible Functions (DFF)* separately on each dimension and show that this method can be used for both two- and three-dimensional bin-packing problems. In a recent paper [8] we propose new lower bounds following the framework defined by Fekete and Schepers. We use a discrete version of DFF and introduce a new class of functions called *Data-Dependent DFF*. These functions behave like DFF for a given instance. The bounds obtained dominate previous