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Inequalities for the ruin probability in a controlled discrete-time risk process

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ABSTRACT

Ruin probabilities in a controlled discrete-time risk process with a Markov chain interest are studied. To reduce the risk of ruin there is a possibility to reinsure a part or the whole reserve. Recursive and integral equations for ruin probabilities are given. Generalized Lundberg inequalities for the ruin probabilities are derived given a constant stationary policy. The relationships between these inequalities are discussed. To illustrate these results some numerical examples are included.

1. Introduction

This paper studies an insurance model where the risk process can be controlled by proportional reinsurance. The performance criterion is to choose reinsurance control strategies to bound the ruin probability of a discrete-time process with a Markov chain interest. Controlling a risk process is a very active area of research, particularly in the last decade; see [8,9,11,12], for instance. Nevertheless obtaining explicit optimal solutions is a difficult task in a general setting even for the classical risk process. In fact, the explicit solution in the classical controlled risk process is known only in a very few cases. This is the main reason why we deal with the *optimal* control problem by using an alternative method than dynamic programming. Hence, an alternative method commonly used in ruin theory is to derive *inequalities* for ruin probabilities (see [1,4–6,12,13]). Following Cai [2] and Cai and Dickson [3], we model the interest rate process as a denumerable state Markov chain. This model can be in fact a discrete counterpart of the most frequently occurring effect observed in continuous interest rate process, e.g., mean-reverting effect. Stochastic inequalities for the ruin probabilities are derived by martingales and inductive techniques. The inequalities can be used to obtain upper bounds for the ruin probabilities. We use the proposal by Gaier et al. [4] and Schmidli [12], we restrict ourselves to use constant control policies (see Remark 1). Explicit condition are obtained for the optimality of employing no reinsurance.

The outline of the paper is as follows. In Section 2 the risk model is formulated. Some important special cases of this model are briefly discussed. In Section 3 we derive recursive equations for finite-horizon ruin probabilities and integral equations for the ultimate ruin probability. In Section 4 we obtain upper bounds for the ultimate probability of ruin. An analysis of the new bounds and a comparison with the Lundberg's inequality is also included. Finally, in Section 5 we illustrate our results on the ruin probability in a risk process with a heavy tail claims distribution under proportional reinsurance and a Markov interest rate process. We conclude in Section 6 with some general comments and some suggestions further research.

2. The model

We consider a discrete-time insurance risk process in which the surplus X_n varies according to the equation

$$X_n = X_{n-1}(1 + I_n) + \mathscr{C}(b_{n-1}) \cdot Z_n - h(b_{n-1}, Y_n), \tag{1}$$

for $n \geqslant 1$, with $X_0 = x \geqslant 0$. Following Schmidli [12, p. 21], we introduce an absorbing (cemetery) state $\varkappa$, such that if $X_n < 0$ or $X_n = \varkappa$, then $X_{n+1} = \varkappa$. We denote the state space by $\mathbb{X} = \mathbb{R} \cup \{\varkappa\}$. Let Y_n be the *nth claim* payment, which we assume to form a sequence of i.i.d. random variables with common probability distribution function

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(p.d.f.) F. The random variable Z_n stands for the *length of the nth period*, that is, the time between the occurrence of the claims Y_{n-1} and Y_n. We assume that $\{Z_n\}$ is a sequence of i.i.d. random variables with p.d.f. G. This case includes a controlled version of the Cramér–Lundberg model if we assume that the claims occur as a Poisson process. Of course, we can also think of the case where $Z_n = 1$ is deterministic. In addition, we suppose that $\{Y_n\}_{n\geqslant 1}$ and $\{Z_n\}_{n\geqslant 1}$ are independent.

The process can be controlled by reinsurance, that is, by choosing the *retention level* (or proportionality factor or risk exposure) $b \in \mathscr{B}$ of a reinsurance contract for one period, where $\mathscr{B} := [b_{\min}, 1]$, and $b_{\min} \in (0,1]$ will be introduced below. Let $\{I_n\}_{n\geqslant 0}$ be the *interest rate process*; we suppose that I_n evolves as a Markov chain with a denumerable (possibly finite) state space $\mathbb{I}$ consisting of non-negative rational numbers and the process $\{I_n\}_{n\geqslant 0}$ is independent of $\{Z_n\}_{n\geqslant 1}$[1] and $\{Y_n\}_{n\geqslant 1}$.

The function $h(b,y)$ with values in $[0,y]$ specifies the fraction of the claim y paid by the insurer, and it also depends on the retention level b at the beginning of the period. Hence $y - h(b,y)$ is the part paid by the reinsurer. The retention level $b = 1$ stands for the control action *no reinsurance*. In this article, we consider the case of *proportional reinsurance*, which means that

$$h(b,y) := b \cdot y, \tag{2}$$

with retention level $b \in \mathscr{B}$. The premium (income) rate c is fixed. Since the insurer pays to the reinsurer a premium rate, which depends on the retention level b, we denote by $\mathscr{C}(b)$ the premium left for the insurer if the retention level b is chosen, with

$$0 \leqslant c_{\min} < c^* \leqslant \mathscr{C}(b) \leqslant c, \quad b \in \mathscr{B},$$

where c^* denotes the minimal value of the premium considered by the insurer and $c_{\min}$ corresponds to $b = 0$ in $\mathscr{C}(b)$. We define $b_{\min} := \min\{b \in (0,1] | \mathscr{C}(b) \geqslant c^*\}$. Since $\mathscr{C}(b) \geqslant c^* > c_{\min} \geqslant 0$, there exists a $b_{\min} > 0$. Moreover, $\mathscr{C}(b)$ is an increasing function that we will calculate according to the *expected value principle* with added safety loading θ from the reinsurer:

$$\mathscr{C}(b) = c - (1+\theta)(1-b)\frac{E[Y]}{E[Z]}, \tag{3}$$

where $E[Y]$ denotes the mean claim and $E[Z]$ denotes the average time between claims. Note that, $c_{\min} = c - (1+\theta)\frac{E[Y]}{E[Z]}$ which we assume that to be greater or equal than zero.

We define Markovian control policies $\pi = \{a_n\}_{n\geqslant 1}$, which at each time n depend only on the current state, that is, $a_n(X_n) := b_n$ for $n \geqslant 0$. Abusing notation, we will identify functions $a : \mathbb{X} \to \mathscr{B}$ with stationary strategies, where $\mathscr{B} = [b_{\min}, 1]$ is the decision space. However, in this study we will focus on stationary constant policies following the arguments in Remark 1. Consider an arbitrary initial state $X_0 = x \geqslant 0$ (note that the initial value is not stochastic) and a control policy $\pi = \{a_n\}_{n\geqslant 1}$. Then, by iteration of (1) and assuming (2), and (3), it follows that for $n \geqslant 1$, X_n satisfies

$$X_n = x\prod_{l=1}^{n}(1+I_l) + \sum_{l=1}^{n}\left((\mathscr{C}(b_{l-1})Z_l - b_{l-1}\cdot Y_l)\prod_{m=l+1}^{n}(1+I_m)\right). \tag{4}$$

Let (p_{ij}) be the matrix of transition probabilities of $\{I_n\}$, i.e.,

$$p_{ij} := P(I_{n+1} = j | I_n = i), \tag{5}$$

where $p_{ij} \geqslant 0$ and $\sum_j p_{ij} = 1$ for all $i,j \in \mathbb{I}$. The ruin probability when using the policy π, given the initial surplus x, and the initial interest rate $I_0 = i$ is defined as

$$\psi^\pi(x,i) := P^\pi\left(\bigcup_{k=1}^{\infty}\{X_k < 0\} | X_0 = x, I_0 = i\right), \tag{6}$$

which we can also express as

$$\psi^\pi(x,i) = P^\pi(X_k < 0 \text{ for some } k \geqslant 1 | X_0 = x, I_0 = i). \tag{7}$$

Similarly, the ruin probabilities in the finite horizon case are given by

$$\psi_n^\pi(x,i) := P^\pi\left(\bigcup_{k=1}^{n}\{X_k < 0\} | X_0 = x, I_0 = i\right). \tag{8}$$

Thus,

$$\psi_1^\pi(x,i) \leqslant \psi_2^\pi(x,i) \leqslant \cdots \leqslant \psi_n^\pi(x,i) \leqslant \cdots,$$

and

$$\lim_{n\to\infty}\psi_n^\pi(x,i) = \psi^\pi(x,i).$$

The following lemma is used below to simplify some calculations.

Lemma 1. *For any given policy* π, *there is a function* $\psi^\pi(x)$ *such that*

$$\psi^\pi(x,i) \leqslant \psi^\pi(x)$$

for every initial state $x > 0$ *and initial interest rate* $I_0 = i$.

Proof. By (1) and (2), the risk model is given by

$$X_n = X_{n-1}(1+I_n) + \mathscr{C}(b_{n-1})Z_n - b_{n-1}Y_n.$$

Since $I_n \geqslant 0$, we have

$$\begin{aligned} X_n &= X_{n-1}(1+I_n) + \mathscr{C}(b_{n-1})Z_n - b_{n-1}Y_n \\ &\geqslant X_{n-1} + \mathscr{C}(b_{n-1})Z_n - b_{n-1}Y_n. \end{aligned} \tag{9}$$

Define recursively

$$\widetilde{X}_n := \widetilde{X}_{n-1} + \mathscr{C}(b_{n-1})Z_n - b_{n-1}Y_n, \tag{10}$$

with $X_0 = \widetilde{X}_0 = x$. Hence, $X_n \geqslant \widetilde{X}_n$ for all $n \in \mathbb{N}$. Clearly, if $X_n < 0$, then $\widetilde{X}_n < 0$.

Let

$$\mathscr{E}_1 := \left\{\omega \in \Omega \,\middle|\, \bigcup_{n=1}^{\infty}\{X_n(\omega) < 0\}\right\} \quad \text{and}$$

$$\mathscr{E}_2 := \left\{\omega \in \Omega \,\middle|\, \bigcup_{n=1}^{\infty}\left\{\widetilde{X_n}(\omega) < 0\right\}\right\},$$

and note that $\mathscr{E}_1 \subset \mathscr{E}_2$. Therefore,

$$P^\pi\left(\bigcup_{n=1}^{\infty}\{X_n < 0\} | I_0 = i\right) \leqslant P^\pi\left(\bigcup_{n=1}^{\infty}\{\widetilde{X}_n < 0\} | I_0 = i\right),$$

and since the $\widetilde{X}_n$ do not depend on I_n, we obtain from (6)

$$\begin{aligned} \psi^\pi(x,i) &= P^\pi\left(\bigcup_{n=1}^{\infty}\{X_n < 0\} | X_0 = x, I_0 = i\right) \\ &\leqslant P^\pi\left(\bigcup_{n=1}^{\infty}\{\widetilde{X}_n < 0\} | X_0 = x\right) =: \psi^\pi(x). \qquad \square \end{aligned}$$

We denote by Π the policy space. A control policy π^* is said to be optimal if for any initial values $(X_0, I_0) = (x,i)$, we have

$$\psi^{\pi^*}(x,i) \leqslant \psi^\pi(x,i)$$

[1] This hypothesis is necessary when one works with a methodology based on martingale theory, as it is the case. Moreover, from the financial point of view, it is not difficult to find particular scenarios for which the interest rate and the waiting time are independent. For example, if one invest in a predictable asset, which is typical on stochastic models of interest rates in order to preserve risk-neutral pricing, and if one considers that the number of events before the maturity may be zero or one, which is not restrictive because these maturity periods are often very short on applications, then, to consider independence between *capitalization* and *waiting time* seems quite natural.