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Convergence of the stationary distributions of $M/M/s/K$ retrial queue as K tends to infinity

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Abstract

We investigate the convergence of the stationary distributions $\boldsymbol{x}(K)$ of $M/M/s/K$ retrial queue to the stationary distribution $\boldsymbol{\pi}$ of $M/M/s$ queue as K tends to infinity. It is showed that $\boldsymbol{x}(K)$ converges geometrically to $\boldsymbol{\pi}$ in l_1-sense and the convergence rate is characterized by the traffic intensity $\rho = \frac{\lambda}{s\mu}$, where λ and μ are the arrival rate and service rate, respectively.

Keywords: $M/M/s/K$ retrial queue; $M/M/s$ queue; Convergence rate; Censored Markov chain

1. Introduction

We consider an $M/M/s/K$ retrial queue with s identical servers and K–s waiting positions. Service times of customers are independent of each other and have a common exponential distribution with parameter μ. Customers arrive according to a Poisson process with rate λ. When an arriving customer finds that all the servers are busy and no waiting position is available, the customer joins a virtual pool of blocked customers called orbit and repeats its request after a random amount of time until the customer gets into the service facility. The capacity of orbit is infinite. The access from orbit to the service facility is governed by the exponential distribution with rate γ_n which may depend on the current number n, $n \geqslant 0$ of customers in orbit. That is, the probability of repeated attempt during the interval $(t, t + \Delta t)$, given that n customers are in orbit at time t, is $\gamma_n \Delta t + o(\Delta t)$. We assume that $\gamma_n \geqslant \gamma_1 = \gamma > 0$, $n \geqslant 1$ and $\gamma_0 = 0$.

From the retrial phenomena, one can intuitively expect that the behavior of the $M/M/s/K$ retrial queue is similar to that of ordinary $M/M/s$ queue while there are no customers in orbit and this situation can occur more frequently as the system is less congested. To reduce the system congestion, at least three ways can be considered. One way is to increase the retrial rate. Some works have been done from this perspective by

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showing that higher retrial rates result in less system congestion (e.g. Liang and Kulkarni, 1993; Shin, 2006) or getting closer to the corresponding ordinary queues (Falin and Templeton, 1997). Falin and Templeton (1997) showed that the stationary distribution of *M*/*M*/*s*/*s* retrial queue (resp. *M*/*G*/*1* retrial queue) with retrial rate $\gamma_n = n\gamma$ converges to that of the ordinary *M*/*M*/*s* queue (resp. $M/G/1/\infty$ queue) as γ tends to infinity and the convergence rate is $O(\frac{1}{\gamma})$. Liang (1999) showed that stochastically longer service times or less servers will result in more customers in the system. From this perspective, one can consider the second way to reduce the system congestion in retrial queue that is to increase the service rates so that the traffic intensity becomes small and hence arriving customers get less chance to be rejected and join orbit. The third way is to increase the capacity *K* in the service facility. By doing so, arriving customers get more chance to get into the service facility and the system performance gets closer to the ordinary queue. In this paper, we investigate the third case, that is, we compare the *M*/*M*/*s*/*K* retrial queue with the ordinary *M*/*M*/*s* queue as *K* tends to infinity. The convergence rate of the stationary distributions of the number of customers in *M*/*M*/*s*/*K* retrial queues is the main concern of this paper.

Convergence of the stationary distribution of finite-capacity systems to that of infinite-capacity system in the class of ordinary queueing systems without retrial phenomena has been investigated by several authors (Choi and Kim, 2000; Choi et al., 2003a; Heyman and Whitt, 1989; Simonot, 1997, 1998). The preceding results in ordinary queues are concerned with the convergence of the systems with finite states, while the convergence of the systems with infinite states are considered in this paper.

This paper is organized as follows. In Section 2, the main result is presented. The main result is proved by comparing with the stationary distributions of some censored Markov chains. The queue length process in *M*/*M*/*s*/*K* retrial queue and its censored Markov chain are investigated in Sections 3 and 4, respectively. In Section 5, *M*/*M*/*s*/*K* retrial queue with constant retrial rate is considered for an upper bound of the probability that there are more than one customers in orbit in the *M*/*M*/*s*/*K* retrial queue. In Section 6, some numerical results are presented and the effects of retrial rates to the speed of convergence are numerically investigated. Finally, concluding remarks are given in Section 7.

2. Main result

Let $X_{K0}(t)$ and $X_{K1}(t)$ represent the number of customers in orbit and in the service facility at time *t*, respectively in *M*/*M*/*s*/*K* retrial queue and $\boldsymbol{X}_K = \{X_K(t), t \geqslant 0\}$ with $X_K(t) = (X_{K0}(t), X_{K1}(t))$. Then the stochastic process $\boldsymbol{X}_K$ is a Markov chain on the state space $\mathscr{S} = \{(i,j) : i = 0,1,2,\ldots,\ j = 0,1,2,\ldots,K\}$. We assume that $\rho = \frac{\lambda}{s\mu} < 1$ which guarantees the existence of the stationary distribution of $\boldsymbol{X}_K$ (He et al., 2000). Let

$$x_{ij}(K) = \lim_{t\to\infty} P(X_{K0}(t) = i, X_{K1}(t) = j), \quad i = 0,1,2,\ldots,\ j = 0,1,2,\ldots,K$$

and $\boldsymbol{x}(K) = (\boldsymbol{x}_0(K), \boldsymbol{x}_1(K), \boldsymbol{x}_2(K), \ldots)$ with $\boldsymbol{x}_n(K) = (x_{n0}(K), \ldots, x_{nK}(K))$, $n \geqslant 0$.

In this section, we present the comparison result between $\boldsymbol{x}(K)$ and the stationary distribution $\boldsymbol{\pi} = \{\pi_j, j \geqslant 0\}$ of *M*/*M*/*s* queue with arrival rate λ and service rate μ. In order to map from the state space $\{0,1,2,\ldots\}$ of *M*/*M*/*s* queue to the state space $\mathscr{S}$ of $\boldsymbol{X}_K$, we consider *M*/*M*/*s* queue as the *M*/*M*/*s*/*K* retrial queue with infinite retrial rate $\gamma_n = \infty$, $n \geqslant 1$. Thus if there are $n > K$ customers in *M*/*M*/*s* queue, then we consider there are $n - K$ customers in orbit and *K* customers in service facility. From this perspective, we define $\tilde{\boldsymbol{\pi}} = \{\tilde{\pi}_{ij}, (i,j) \in \mathscr{S}\}$ as

$$\tilde{\pi}_{ij} = \begin{cases} \pi_j, & i = 0, \quad 0 \leqslant j \leqslant K, \\ \pi_{i+K}, & i \geqslant 1, \quad j = K, \\ 0, & i \geqslant 1, \quad 0 \leqslant j \leqslant K-1 \end{cases}$$

and compare $\boldsymbol{x}(K)$ with $\boldsymbol{\pi}$ using $\tilde{\pi}$ by the following formula:

$$\|\boldsymbol{\pi} - \boldsymbol{x}(K)\| \triangleq \sum_{i=0}^{\infty}\sum_{j=0}^{K} |\tilde{\pi}_{ij} - x_{ij}(K)| = \sum_{j=0}^{K} |\pi_j - x_{0j}(K)| + \sum_{i=1}^{\infty} |\pi_{i+K} - x_{iK}(K)| + \sum_{i=1}^{\infty}\sum_{j=0}^{K-1} x_{ij}(K).$$