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Highway games on weakly cyclic graphs ☆

Barış Çiftçi *, Peter Borm, Herbert Hamers

CentER and Department of Econometrics and Operations Research, Tilburg University, P.O. Box 90153, 5000 LE, Tilburg, The Netherlands

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ABSTRACT

A *highway problem* is determined by a connected graph which provides all potential entry and exit vertices and all possible edges that can be constructed between vertices, a cost function on the edges of the graph and a set of players, each in need of constructing a connection between a specific entry and exit vertex. Mosquera (2007) introduce *highway problems* and the corresponding cooperative cost games called *highway games* to address the problem of fair allocation of the construction costs in case the underlying graph is a tree. In this paper, we study the concavity and the balancedness of highway games on weakly cyclic graphs. A graph G is called highway-game concave if for each highway problem in which G is the underlying graph the corresponding highway game is concave. We show that a graph is highway-game concave if and only if it is weakly triangular. Moreover, we prove that highway games on weakly cyclic graphs are balanced.

1. Introduction

Mosquera (2007) address the problem of fair allocation of the construction costs of a highway network. For this aim, they formally consider highway problems and analyze the corresponding cooperative cost games called highway games. In a highway problem, the possibilities regarding the construction of the highway network are determined by a connected graph. The set of vertices of the graph represents the potential entry and exit points and the edges in the graph represent the possible highway connections that can be constructed. Each edge in the graph has an associated cost which in general will depend on its length or the geographical properties that may affect the construction costs of the highway. Each player in a highway problem has to establish a connection between two vertices in the graph, i.e., between his entry and exit point. Given a highway problem, a corresponding highway game is defined as a cooperative cost game which associates to each coalition of players the total cost of the cheapest selection of edges in the graph which connects the entry and exit point of every member of the coalition. Mosquera (2007) restricted attention to highway problems in which the underlying graph is a tree. In this setting, there is only one path between an entry and exit point.

In this paper, we study highway problems in which the underlying graphs are weakly cyclic, i.e., connected graphs for which every edge in the graph is contained in at most one cycle. In particular, these graphs may contain cycles and hence, there will exist multiple paths between some entry and exit points. Note that, in this setting, a coalition of players can further reduce the joint construction costs by an optimal coordination of paths to construct. That is, the joint minimal cost of a coalition is now obtained as a result of solving a combinatorial optimization problem. Hence highway games induced by weakly cyclic graphs belong to the research area of operations research games which focuses on the interplay between the optimization of costs of a project and the allocation of costs among the participants of the project. From among the numerous studies on this topic, we mention minimum cost spanning tree games (Granot and Huberman, 1981), traveling salesman games (Potters et al., 1992), Chinese postman games (Granot et al., 1999), sequencing games (Curiel et al., 1989) and project games (Estevez-Fernandez et al., 2007). An overview of operations research games can be found in Borm et al. (2001).

We start our analysis of highway games by investigating their concavity properties. A cooperative cost game is called concave if it exhibits the property that the incentives to join a coalition increase as the coalition becomes larger. We proceed as Herer and Penn (1995) on traveling salesman problems and Granot et al. (1999) on Chinese postman problems, and focus on the question for which class of graphs the corresponding games are always concave. We define a graph to be highway game-concave (HG-concave), if for every player set, for

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* Corresponding author. Tel.: +31 63 376 7536; fax: +31 13 466 3280.
E-mail address: b.b.ciftci@uvt.nl (B. Çiftçi).

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every choice of entry and exit points for the players and for every cost specification, the corresponding highway game is concave. We show that a graph is HG-concave if and only if it is weakly triangular. Here, a graph is called weakly triangular if it is weakly cyclic and moreover, if every cycle is a triangle, i.e., if every cycle is composed of precisely three edges.

We then investigate the core of the highway games induced by weakly cyclic graphs. The core of a cost game is defined as the set of cost allocations that are stable in the sense that no coalition of players can do better by splitting off. For general highway games in which the underlying graphs allow for multiple paths between vertices, the core can be empty. In this paper, we prove that highway games induced by weakly cyclic graphs have non-empty cores.

Most of the current literature on the allocation of the construction costs of networks focuses on variants of minimum cost spanning tree (mcst) problems. These problems typically consider a group of players, each of whom has to be connected to a source, either directly or via other players. The main difference between highway problems and mcst problems is that, in a highway problem, there is no particular vertex every player has to be connected to. However, highway problems are a special type of minimum cost forest (mcf) problems as introduced by Kuipers (1997), but only if the underlying graph is complete. Mcf problems are generalizations of mcst problems which allow for more than one source, where each source offers a different type of service and each customer has to be connected with a non-empty subset of the available sources.

The outline of the paper is as follows. Section 2 recalls basic notions from cooperative game theory and graph theory and formally introduces highway problems and highway games. Section 3 provides the characterization of HG-concave graphs. Section 4 proves the balancedness of highway games on weakly cyclic graphs.

2. Highway problems and highway games

In this section, we formally define highway problems and the corresponding cooperative cost games, called highway games.

2.1. Preliminaries

A cooperative (cost) game is a pair (N,c), where N is a non-empty, finite set of players and c is a mapping, $c\,2^N \to \mathbb{R}$ with $c(\emptyset) = 0$. A *coalition* is a set of players $S \subset N$ and N is called *the grand coalition*. For any coalition $S \subset N$, $c(S)$ is interpreted as the minimal joint cost of coalition S. A game (N,c) is *monotonic* if $c(S) \geqslant c(T)$ for every $S,T \in 2^N$ with $T \subset S$ and it is called *subadditive* if $c(S) + c(T) \geqslant c(S \cup T)$ for every $S,T \in 2^N$ with $T \cap S = \emptyset$. A game (N,c) is *concave* if $c(T \cup S) + c(T \cap S) \leqslant c(T) + c(S)$ for every $S,T \subset N$.

A cost allocation $x \in \mathbb{R}^N$ for a game (N,c) is called *efficient* if $\sum_{i \in N} x_i = c(N)$. The *core* Core$(c)$ of a game (N,c) is defined as the set of efficient cost allocations for which no coalition has an incentive to split off from the grand coalition, i.e.,

$$\mathrm{Core}(c) = \left\{ x \in \mathbb{R}^N | \sum_{i \in N} x_i = c(N) \text{ and } \sum_{i \in S} x_i \leqslant c(S) \text{ for all } S \in 2^N \right\}.$$

A *balanced set* $\mathcal{B}$ is a collection of subsets S of N with the property that there exist positive numbers $\lambda_S,\ S \in \mathcal{B}$, called *weights*, such that for each $i \in N$, we have that

$$\sum_{S \in \mathcal{B}\, i \in S} \lambda_S = 1.$$

A game (N,c) is called *balanced* if

$$\sum_{S \in \mathcal{B}} \lambda_S c(S) \geqslant c(N),$$

for every balanced set $\mathcal{B}$ and for every corresponding set of weights $(\lambda_S)_{S \in \mathcal{B}}$. It is well known that a game has a non-empty core if and only if it is balanced. It is also well known that concave games are balanced.

A *graph* G is a pair (V,E), where V is a non-empty and finite set of *vertices* and $E \subset E_V = \{\{u,v\} | u,v \in V, u \neq v\}$ is a set of *edges*. (V,E_V) is called the *complete graph* on V. A *subgraph* of $G = (V,E)$ is a graph $G' = (V',E')$ with $\emptyset \neq V' \subset V$ and $E' \subset E_{V'} \cap E$. If G' is a subgraph of G, then we say that G *contains* G'.

A *path* P of length $k \geqslant 1$ is a graph (V,E) with

$$V = \{v_1, \ldots, v_{k+1}\}, \quad |V| = k+1 \text{ and } E = \{\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_k,v_{k+1}\}\},$$

and is often abbreviated by $P = v_1, v_2, \ldots, v_{k+1}$; it is also referred as *a path from* v_1 to v_{k+1}.

A *cycle* C of length $k \geqslant 3$ is a graph (V,E) with

$$V = \{v_1, \ldots, v_k\}, \quad |V| = k \quad \text{and} \quad E = \{\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_{k-1},v_k\},\{v_k,v_1\}\},$$

and is often abbreviated by $C = v_1, v_2, \ldots, v_k v_1$.

Two vertices $u,v \in V$ are called *connected* in G if $u = v$ or if G contains a path between u and v as a subgraph. A graph G is called *connected* if every pair of distinct vertices of G is connected in G. If $G = (V,E)$ is connected, an edge $e \in E$ is called a *bridge* in G if the graph $(V, E \setminus \{e\})$ is not connected. A connected graph is called *weakly cyclic* if every edge in the graph is contained in at most one cycle.[1] A weakly cyclic graph is called *weakly triangular* if every cycle in G has length 3.

[1] The name "weakly cyclic" graph is not a standard graph theoretical term. It was first introduced in the context of Chinese postman games in Granot et al. (1999) and was maintained in subsequent related papers. Weakly cyclic graphs are also called *cactus graphs*.