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Discrete Optimization

Locating an obnoxious plane

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Abstract

Let S be a set of n points in three-dimensional Euclidean space. We consider the problem of positioning a plane π intersecting the convex hull of S such that $\min\{d(\pi,p); p \in S\}$ is maximized. In a geometric setting, the problem asks for the widest empty slab through n points in space, where a slab is the open region of $\mathbb{R}^3$ that is bounded by two parallel planes that intersect the convex hull of S. We give a characterization of the planes which are locally optimal and we show that the problem can be solved in $O(n^3)$ time and $O(n^2)$ space. We also consider several variants of the problem which include constraining the obnoxious plane to contain a given line or point and computing the widest empty slab for polyhedral obstacles. Finally, we show how to adapt our method for computing a largest empty annulus in the plane, improving the known time bound $O(n^3 \log n)$ [J.M. Díaz-Báñez, F. Hurtado, H. Meijer, D. Rappaport, T. Sellarès, The largest empty annulus problem, International Journal of Computational Geometry and Applications 13 (4) (2003) 317–325].

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1. Introduction

Location science is a classical field of operations research that has also been considered in the computational geometry community. A class of problems from this field, often referred to as maximin facility location, deals with the placement of undesirable or obnoxious facilities. In these problems the objective is to maximize the minimal distance between the facility and a set of input points. Furthermore, in order to ensure that the problems are well-defined, the facility is normally constrained to go through some sort of bounding region, such as the convex hull or bounding box of the input

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points. Applications of these problems go well beyond the field of location science. For instance, splitting the space using cuts that avoid the input points is useful in areas like cluster analysis, robot motion-planning and computer graphics.

Maximin facility location problems have recently been considered in computational geometry. Maximin criteria have been investigated in 2-d for the optimal positioning of points [25,4], lines [13], anchored lines [12], and circumferences [8]. When the facility is a line, the problem is equivalent to that of computing a *widest empty corridor*, i.e., a largest empty open space bounded by two parallel lines. Variants of the problem have also been considered and include corridors containing k input points [16,22,6], dynamic updates [16,22] and L-shaped corridors [7]. Most of the results to date are two-dimensional and, with a few exceptions (e.g., [12]), little progress has been reported in three dimensions. In a recent work [14], a bichromatic separating problem has been solved in three dimensions.

The most classical versions of facility location problems consider the positioning of one or several point-like facilities. Nowadays, there is a growing body of research on the location of non-point facilities. For example, line location problems have been extensively studied both in the plane [17,23] and in the three-dimensional space [24,5]. See [9] for a recent survey on the current state-of-art of these problems. In this paper, we deal with the maximin location of a plane in 3-d. We formulate the *obnoxious plane problem*, **OPP**, as follows: Given a set S of n points in $\mathbb{R}^3$, find a plane π intersecting the convex hull of S which maximizes the minimum Euclidean distance to the points.

Notice that, in 2-d, our problem reduces to that of computing the *widest empty corridor* through a set of points in the plane. This problem has been solved in $O(n^2)$ time and $O(n)$ space [13]. We extend the definition of corridor through a point set from $\mathbb{R}^2$ to $\mathbb{R}^3$ as follows: a *slab* through S is the open region of $\mathbb{R}^3$ that is bounded by two parallel planes that intersect the convex hull of S. The *width* of the slab is the distance between the bounding planes. Thus, we are interested in finding *the widest empty slab*.

It is natural to consider a "dual" version of our problem, where the goal is to minimize the maximal distance between the plane and a set of input points. This minimax location problem was solved in [15] for 3-d using techniques different from our own. Even for 2-d, the approaches used to solve the minimax [18] and the maximin [13] versions are very different.

The rest of the paper is organized as follows. In Section 2, we present some notation and preliminary results. In Section 3, we describe an algorithm to compute an obnoxious plane in $O(n^3)$ time and $O(n^2)$ space. Other variants, obtained by constraining the optimal plane π to go through a given line or given point, are described in Section 4, and solved in $O(n \log n)$ and $O(n^{2+\varepsilon})$ time, respectively. In Section 5, we compute the widest empty slab through a set of polyhedral obstacles within the same bounds as the **OPP**. Section 6 presents a reduction of the largest empty annulus problem to our problem. Finally, Section 7 contains some concluding remarks and open problems.

2. Characterization of candidate planes

In this section we describe a simple formula to compute the width of a slab and derive necessary conditions for slab optimality.

Observation 1. Let π and σ be two distinct parallel planes with (common) unit normal $\vec{n}$. Let p and q be arbitrary points on π and σ, respectively. Then, $\mathrm{dist}(\pi, \sigma) = |\vec{n} \cdot (q - p)|$.

The following lemma characterizes candidate solutions for the **OPP**,

Lemma 1. *Let π^* be a solution to an instance of* **OPP** *and let π_1 and π_2 be the bounding planes of the slab generated by π^*. Then, exactly one of the following conditions must hold:*

(a) *Each of π_1 and π_2 contains exactly one point of S, p_1 and p_2 respectively, such that $p_2 - p_1$ is orthogonal to π^*.*
(b) *There are points $S_1 = \{p_{11}, \ldots, p_{1h}\} \subset S$ on π_1 and $S_2 = \{p_{21}, \ldots, p_{2k}\} \subset S$ on π_2 such that $h \geqslant 2$, $k \geqslant 1$ and $S_1 \cup S_2$ lie on a common plane τ that is orthogonal to π^*.*