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Discrete Optimization

A level-2 reformulation–linearization technique bound for the quadratic assignment problem

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Abstract

This paper studies polyhedral methods for the quadratic assignment problem. Bounds on the objective value are obtained using mixed 0–1 linear representations that result from a reformulation–linearization technique (rlt). The rlt provides different "levels" of representations that give increasing strength. Prior studies have shown that even the weakest level-1 form yields very tight bounds, which in turn lead to improved solution methodologies. This paper focuses on implementing level-2. We compare level-2 with level-1 and other bounding mechanisms, in terms of both overall strength and ease of computation. In so doing, we extend earlier work on level-1 by implementing a Lagrangian relaxation that exploits block-diagonal structure present in the constraints. The bounds are embedded within an enumerative algorithm to devise an exact solution strategy. Our computer results are notable, exhibiting a dramatic reduction in nodes examined in the enumerative phase, and allowing for the exact solution of large instances.

Keywords: Combinatorial optimization; Assignment; Branch and bound; Quadratic assignment problem; Reformulation–linearization technique

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1. Introduction

The standard mathematical formulation of the quadratic assignment problem is as follows

$$\text{QAP}: \quad \min\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}x_{ij} + \sum_{\substack{i=1\\ i\neq k}}^{n}\sum_{\substack{j=1\\ j\neq l}}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} C_{ijkl}x_{ij}x_{kl} : \mathbf{x}\in\mathbf{X}, \mathbf{x} \text{ binary}\right\}, \tag{1}$$

where

$$\mathbf{x}\in\mathbf{X} \equiv \left\{\mathbf{x} \geqslant \mathbf{0} : \sum_{i=1}^{n} x_{ij} = 1 \quad \text{for } j = 1,\ldots,n;\ \sum_{j=1}^{n} x_{ij} = 1 \quad \text{for } i = 1,\ldots,n\right\}. \tag{2}$$

The problem is so named because the objective is to optimize a quadratic function of binary variables over the assignment polytope $\mathbf{X}$. The objective contains no quadratic expressions $x_{ij}x_{kl}$ having $i = k$ or $j = l$ since the $\mathbf{x}$ binary restrictions force $x_{ij}x_{kl} = x_{ij}$ if $i = k$ and $j = l$, and the $\mathbf{x}$ binary and assignment restrictions together force $x_{ij}x_{kl} = 0$ otherwise. We use the abbreviations qap and QAP throughout the paper to refer to "quadratic assignment problem" and problem QAP above, respectively. For notational convenience, we henceforth let all summations run from 1 to n unless noted otherwise.

The qap is among the most difficult NP-hard combinatorial optimization problems. In theory, it can be solved by enumerating the n factorial feasible binary solutions, and by selecting one that yields a minimal value. But from a practical point of view, it is extremely challenging, with exact procedures tending to fail for problem sizes of about $n = 25$ to $n = 30$, i.e. for 625–900 variables.

Among exact methods, branch-and-bound approaches have been the most successful. Here, the intent is to implicitly enumerate over the set of solutions, using lower bounds to prune branches of the binary search tree. The key challenge has been to obtain tight bounds that permit effective pruning and that are not too expensive to compute. Interestingly, the qap has proven itself more challenging than other classes of NP hard problems in terms of the size instances that can be solved. A partial explanation is that the majority of test problems suffer from a homogeneous objective function which tends to hurt the pruning process.

Our prior research on the qap has led to computational advances, and pointed the way for the current study. These earlier efforts were based on the application of a reformulation–linearization-technique (rlt) to QAP. The rlt recasts QAP as a mixed 0–1 linear program via two steps. It first *reformulates* the problem by constructing redundant nonlinear restrictions, obtained by multiplying the equality constraints of $\mathbf{X}$ by product factors of the binary variables. Thereafter, it *linearizes* the objective and constraints by substituting a continuous variable for each distinct nonlinear term. Depending on the product factors used to compute the redundant restrictions, different formulations emerge. The result is an n-level hierarchy of mixed 0–1 linear representations of QAP. Each level of the hierarchy provides a program whose continuous relaxation is at least as tight as the previous level, with the highest level giving a convex hull representation.

The *weakest* level-1 rlt form, which follows from the work of [2,3], was shown [1,14] to subsume and unify alternate linear representations of QAP, and the resulting bounds to dominate the majority of published works in terms of relaxation strength. In addition, this form has a block-diagonal structure [1] that lends itself to efficient solution methods; in particular, to a Lagrangian relaxation with special structure in the subproblem as well as the dualized constraints. (Also see [10] for a different interpretation of this same decomposition and bound.) For QAP of size n, the subproblems consist of $n^2 + 1$ separate linear assignment problems, n^2 of size $n - 1$ and one of size n. The dualized equality constraints essentially set one family of variables equal to another, so that each such restriction has exactly two nonzero entries, one 1 and one -1. The overall approach motivates a monotonic increasing sequence of lower bounds, and is referred to as a *dual ascent strategy*.

Bounds from the level-1 rlt were strategically implemented within enumerative algorithms [11–13], resulting in marked success. Well-known test problems up to size $n = 22$ were solved in [11]. Other problems solved include [12] the size $n = 25$ instance from [16], and the Krarup 30a [13]. References to these classical test cases are found in QAPLIB [7]. The only competitive methods of which we are aware are due to Brixius and Anstreicher [6] and Anstreicher et al. [5], which use convex quadratic programming bounds relaxations as in [4].

Based on our successes with the level-1 rlt representation, we turn attention in this paper to the level-2 form. This program provides even tighter bounds than level-1, but at the price of increased size. The challenge is to