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Continuous Optimization

# A smoothing heuristic for a bilevel pricing problem

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**Abstract**

In this paper, we provide a heuristic procedure, that performs well from a global optimality point of view, for an important and difficult class of bilevel programs. The algorithm relies on an interior point approach that can be interpreted as a combination of smoothing and implicit programming techniques. Although the algorithm cannot guarantee global optimality, very good solutions can be obtained through the use of a suitable set of parameters. The algorithm has been tested on large-scale instances of a network pricing problem, an application that fits our modeling framework. Preliminary results show that on hard instances, our approach constitutes an alternative to solvers based on mixed 0–1 programming formulations.


*Keywords:* Global optimization; Bilevel programming; Interior point methods; Smoothing; Implicit programming; MPEC; Complementarity; Network pricing

## 1. Introduction

Our aim in this work is to design an efficient algorithm for solving a class of bilevel programs upon which pricing and revenue management problems have been based (see [9,22,23]). Bilevel programming, or mathematical programming with equilibrium constraints (MPEC), is a branch of mathematics concerned with the optimization of an objective function over the solution set of some mathematical program [32,33]. It

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is closely related to Stackelberg games, where a leader incorporates within her decision process the reaction of the followers to her course of action. It can be fairly argued that all decision-making processes involving variables that are not in the direct control of the leader fit this framework. Examples of such instances abound in industry or government, where the impact of policies on customers should be evaluated before they are implemented. Its mathematical formulation is:

$$(BP) \quad \min_{(u,v)} F(u,v)$$
$$\begin{cases} u \in U \subseteq \mathbb{R}^{n_u}, \\ v \in \arg \min\limits_{v \in V(u) \subseteq \mathbb{R}^{n_v}} f(u,v), \end{cases}$$

where $u$ and $v$ are respectively, the upper and lower-level variables. It is well known that bilevel programming is intractable (see e.g. [19,34,1]), yielding different research avenues. One trend focuses on exact algorithms. Indeed, it is sometimes possible to transform a bilevel program into a (one-level) mixed 0–1 program (see e.g. [1]). Although this has the advantage that general-purpose solvers can be used, this framework tends to break down on large instances.

Another line of attack consists in replacing the follower's optimization problem, whenever it is convex, by its stationarity conditions. This yields a one-level program with complementarity constraints, known as MPEC [25]. The complementarity constraints, that hide the combinatorial structure of the problem, are particularly difficult to handle. Moreover, MPECs are ill-structured in the sense that they generally satisfy no constraint qualification, and that stationarity can fail to be characterized by a Karush–Kuhn–Tucker system, and that general-purpose NLP solvers may fail to uncover such local minima or even mere stationary points.[1] Different approaches have been developed to solve MPECs, and we mention three that are related to our line of attack. First, the implicit programming approach (see e.g. [26,25,11]) can be applied when the follower's choice is unique for every decision made by the leader. It usually leads to the use of nonsmooth analysis techniques [8]. Next, in the smoothing approach [12,16,17,20,30], one reformulates the complementarity constraints as nondifferentiable constraints and then applies smoothing techniques on these constraints.[2] A third approach is provided by the algorithm PIPA of Luo et al. [25], which is based on interior-point methods (see however [24]). From a practical point of view, one drawback of these techniques is that they guarantee local optimality, at best, under a variety of strong assumptions.

Bilevel programming is particularly suited at modeling pricing problems. In this paper, we focus on a network pricing problem (MaxToll in the sequel) that has been defined and analyzed by Labbé et al. [22,23]. In this model, the follower consists in travelers moving between their respective origin and destination, using only shortest paths. At the upper level, the leader has the power to levy tolls on a subset of road segments, and aims at maximizing his revenue. The leader's dilemma is to avoid tolls too low—because they produce low revenue—, and tolls too high—because they urge the network users to choose toll-free paths. It was recently shown that MaxToll, even in its simplest form, is strongly NP-hard [27].[3]

Besides toll optimization, variants of the basic model can be useful for pricing purposes in the airline and telecommunication industries [9]. See also [2] for a related model with applications to tax credits in biofuel production. Actually, our algorithm can be applied to a much larger class of problems, i.e., bilevel programs with bilinear objectives and linear constraints.

In this paper, our aim is to combine the implicit programming and smoothing techniques, and design an algorithm that is both efficient on large realistic instances and performs well from a global optimality point of view. Our choice to study the special case of MaxToll has been motivated both by its practical

[1] Note however that recent attempts at tackling MPECs with general NLP solvers such as SQP solvers [14,15] or interior-point solvers [3] have been successful, to some extent.

[2] This technique originated in works on complementarity problems. For references, see [5,13].

[3] The classic reference on NP-hardness is [18].