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On the distribution of Weierstrass points on Gorenstein quintic curves

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Abstract This paper is concerned with developing a technique to compute in a very precise way the distribution of Weierstrass points on the members of any 1-parameter family C_a, $a \in \mathbb{C}$, of Gorenstein quintic curves with respect to the dualizing sheaf $\mathcal{K}_{C_a}$. The nicest feature of the procedure is that it gives a way to produce examples of existence of Weierstrass points with prescribed special gap sequences, by looking at plane curves or, more generally, to subcanonical curves embedded in some higher dimensional projective space.

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1. Introduction

At the beginning, several researchers developed the theory of the Weierstrass points for smooth curves, and for their canonical divisors. During the last three decades, Lax and Widland (see [1–6]) founded and developed the theory for Gorenstein curves, where the invertible dualizing sheaf replaces the canonical sheaf. Through this context, the singular points of a Gorenstein curve have to be considered as Weierstrass points.

The goal of this paper is to develop a technique for computing the distribution of the Weierstrass points on the members of any 1-parameter family C_a, $a \in \mathbb{C}$, of Gorenstein quintic curves with respect to the dualizing sheaf $\mathcal{K}_{C_a}$. Such a technique is based on the computation of the sequence of integers which in [7] has been called "$\mathcal{K}_{C_a}$-Weierstrass Gaps Sequence" ($\mathcal{K}_{C_a}$-WGS for brief), even at singular points. In [9], the first author and F. Sakai classified and investigated the distribution of Weierstrass points on certain 1-parameter family of genus 3 curves, named after Kuribayashi quartic curves.

Actually, the technique we describe, consists of performing a fixed sequence of computations, and so, it can be applied to any Gorenstein quintic curve, at any point P, no matter if it is smooth or singular. In case, P is smooth, the $\mathcal{K}_{C_a}$-WGS are computed by determining the dimension of the linear systems

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$\mathcal{K}_{C_a} - nP$, for every non-negative integer n, and so some contact order must be computed. If P is singular, the $\mathcal{K}_{C_a}$-WGS are given by a suitable combination of the $\tilde{\mathcal{K}}_{\tilde{C}_a}$-WGS at the points $Q_1, \ldots, Q_m$, $m \geq 1$, over P in a partial normalization $\theta_P : \tilde{C}_a \longrightarrow C_a$ of C_a at P, where $\tilde{\mathcal{K}}_{\tilde{C}_a}$ is the pull back of $\mathcal{K}_{C_a}$. The $\tilde{\mathcal{K}}_{\tilde{C}_a}$-WGS at these points can be computed as the contact order of C_ω, $\omega \in \mathcal{K}_{C_a}$, and the branches $C_a^{(1)}, \ldots, C_a^{(m)}$ of C_a through P, corresponding to $Q_1, \ldots, Q_m$, respectively, one branch at a time. Moreover, the study of the branches through P allows to largely simplify the computation of the $\mathcal{K}_{C_a}$-WGS. This simplification is essentially due to the knowledge of the normalization map in terms of blow-up's, as shown in [7].

In both cases, the contact orders are computed by means of the osculating conics. Moreover, in the next section, we describe a quick way to compute the osculating conics at a point of a Gorenstein quintic curve, because in the most spread computer algebra systems there is no built in function to perform that computation. However, as a computing support, to perform the computations described through this paper we use MATHEMATICA and MAPLE programs.

The layout of the paper is as follows. In Section 2, we cover most of the necessary background material. Section 3 is devoted to describe the technique and show its correctness, while, in the last Section 4, we let the technique work on some interesting examples.

2. Notation and preliminaries

We begin by stating the basic tools that will be used throughout this paper.

2.1. *Weierstrass points*

Here, we briefly recall what we need about Weierstrass points on curves. We start by the definition of the $\mathcal{K}_C$-Weierstrass gap sequences (shortly $\mathcal{K}_C$-WGS in the following) at a point, singular or not, with respect to the dualizing sheaf $\mathcal{K}_C$ over C. To this purpose, let C be any projective integral curve of arithmetic genus g over the complex field $\mathbb{C}$. Let us recall a geometrical definition of a $\mathcal{K}_C$-gap at a point P of C (see [8], §2). If the point P is non-singular, the $\mathcal{K}_C$-WGS at P is defined as follows:

Definition 1. Let P be a smooth point on the curve C. The integer n is a $\mathcal{K}_C$-gap if and only if, $\dim_{\mathbb{C}}(\mathcal{K}_C - (n-1)P) > \dim_{\mathbb{C}}(\mathcal{K}_C - nP)$. The sequence of the $\mathcal{K}_C$-gaps is the $\mathcal{K}_C$-WGS at P.

On the other hand, if $P \in C$ is a singular point, let $\pi : \tilde{C} \longrightarrow C$ is the normalization of C and consider the linear system $\tilde{V} = span(\pi^* v_1, \ldots, \pi^* v_g)$ over $\tilde{C}$, where $(v_1, \ldots, v_g)$ is a basis of $H^0(C, \mathcal{K}_C)$.

A positive integer $b(Q)$ is called a $\tilde{V}$-gap at a point $Q \in \pi^{-1}(P)$ if and only if, $\dim_{\mathbb{C}}(\tilde{V} - (b(Q)-1)Q) > \dim_{\mathbb{C}}(\tilde{V} - b(Q)Q)$. Since $\dim_{\mathbb{C}}(\tilde{V}) = g$ and by Riemann–Roch Theorem $\dim_{\mathbb{C}}(\tilde{V} - (2g-1)Q) = 0$, it follows that at each $Q \in \pi^{-1}(P)$ there are exactly g $\tilde{V}$-gap. If $\tilde{V}$-WGS is known for each point Q lying over P, the $\mathcal{K}_C$-WGS $\{a_1(P), \ldots, a_g(P)\}$ at P can be computed as follows:

Proposition 1. *Suppose $\pi : \tilde{C} \longrightarrow C$ is the normalization of C. Let $Q_1, \ldots, Q_m$ be the points of $\tilde{C}$ corresponding to the branches centered at a point P of C and $\{b_1^{\tilde{V}}(Q_i), \ldots, b_g^{\tilde{V}}(Q_i)\}$ be the $\tilde{V}$-WGS at the point Q_i, for $i = 1, 2, \ldots, m$, then one has:*

$$a_k(P) = \sum_{i=1}^{m} b_k^{\tilde{V}}(Q_i) - k(m-1), \quad 1 \leq k \leq g. \tag{1}$$

Proof. See ([7], Proposition 5.5, p. 285). □

Following [7], Proposition 5.4, one can define the so called kth $\mathcal{K}_C$-extraweight at the point P, denoted by $E_k(P)$ as:

$$E_k(P) = \sum_{Q \in \pi^{-1}(P)} w_k^{\tilde{V}}(Q),$$

where $w_k^{\tilde{V}}(Q) = \sum_{i=1}^{k}(b_k^{\tilde{V}}(Q) - i)$ is the kth $\tilde{V}$-Weierstrass weight at the smooth point Q. Therefore, at the point P, one can attach a sequence of integers $\{E_1(P), \ldots, E_g(P)\}$, called the $\mathcal{K}_C$-extraweight sequence at P. By means of the extraweight sequence, the $\mathcal{K}_C$ -WGS $\{a_1(P), \ldots, a_g(P)\}$ at P can be computed as

$$a_k(P) = \begin{cases} E_k(P) + 1 & \text{if} \quad k = 1, \\ E_k(P) - E_{k-1}(P) + k & \text{if} \quad 2 \leq k \leq g. \end{cases} \tag{2}$$

Hence, we also have (see [7])

$$E_k(P) = \sum_{i=1}^{k}(a_i(P) - i). \tag{3}$$

The last two formulas show that it is equivalent to know the $\mathcal{K}_C$-WGS or the extraweight sequence at P. It is clear that the first way is easier to compute than the second, because of the geometrical meaning of the $\mathcal{K}_C$-WGS.

Using a Widland–Lax argument (see [1] and [6]) or (see [7], Proposition 4.5) one can show that for each k

$$w_k(P) = k(k-1)\delta_P + E_k(P), \quad 1 \leq k \leq g \tag{4}$$

where $w_k(P)$ is a non-negative integer, called kth $\mathcal{K}_C$ -weight at the point P and $\delta_P = \dim_{\mathbb{C}}(\tilde{\mathcal{O}}_P(C)/\mathcal{O}_P(C))$ is a numerical invariant linked to the kind of singularity. The sequence of integers $\{w_1(P), \ldots, w_g(P)\}$ is called the $\mathcal{K}_C$-weight sequence at P. The gth $\mathcal{K}_C$-weight $w_g(P)$ is nothing but the vanishing order at the point P of the Wronskian of a basis for $H^0(C, \mathcal{K}_C)$ as defined in [6]. Hence, the point P is a $\mathcal{K}_C$-Weierstrass point if and only if $w_g(P) > 0$. Moreover, the total number of the $\mathcal{K}_C$-Weierstrass points up to their weights is given by the following proposition (see [6], Proposition 1 or [7], Proposition 4.4).

Proposition 2. *The total gth $\mathcal{K}_C$ -weight of the $\mathcal{K}_C$ -Weierstrass point is:*

$$W_{C,g} = \sum_{P \in C} w_g(P) = (g-1)g(g+1) \tag{5}$$

Remark 1. As a consequence to Proposition 1, if $\pi^{-1}(P) = \{Q\}$, i.e. the preimage of P reduces to just one point on C, then the $\mathcal{K}_C$-WGS at P coincides with the $\tilde{V}$-WGS at Q. The $\mathcal{K}_C$-WGS and the $\tilde{V}$-WGS coincide also when the point P is smooth.