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Original Article

Some constraints of hypergeometric functions to belong to certain subclasses of analytic functions

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Abstract The purpose of this paper is to introduce sufficient conditions for (Gaussian) hypergeometric functions to be in various subclasses of analytic functions. Also, we investigate several mapping properties involving these subclasses.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic and univalent in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. For $g(z) \in \mathcal{A}$ of the form:

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n, \tag{2}$$

the Hadamard product (or convolution) of two power series $f(z)$ and $g(z)$ is given by (see [1]):

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n g_n z^n = (g * f)(z). \tag{3}$$

We recall some definitions which will be used in our paper.

Definition 1.1. For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and written $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$

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$(z \in \mathbb{U})$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence (see [2]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 1.2. A function $f(z) \in \mathcal{A}$ is called starlike of order α (denoted by $\mathcal{S}^*(\alpha)$), if $f(z)$ satisfies the following condition:

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \ (0 \leq \alpha < 1;\ z \in \mathbb{U}). \tag{4}$$

Also, a function $f(z) \in \mathcal{A}$ is called convex of order α (denoted by $\mathcal{K}(\alpha)$), if $f(z)$ satisfies the following condition:

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \ (0 \leq \alpha < 1;\ z \in \mathbb{U}). \tag{5}$$

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were studied by MacGregor [3], Schild [4], Pinchuk [5] and others. From (4) and (5) we can see that

$$f(z) \in \mathcal{K}(\alpha) \Longleftrightarrow zf'(z) \in \mathcal{S}^*(\alpha). \tag{6}$$

We denote by $\mathcal{S}^* = \mathcal{S}^*(0)$ and $\mathcal{K} = \mathcal{K}(0)$, where $\mathcal{S}^*$ and $\mathcal{K}$ are the classes of starlike and convex functions, respectively, (see Robertson [6]).

Definition 1.3. A function $f(z) \in \mathcal{A}$ is said to be k-uniformly convex function (denoted by $k - \mathcal{UCV}$), if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq k\left|\frac{zf''(z)}{f'(z)}\right| \ (k \geq 0;\ z \in \mathbb{U}). \tag{7}$$

Also, a function $f(z) \in \mathcal{A}$ is said to be k -starlike function (denoted by $k - \mathcal{ST}$), if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq k\left|\frac{zf'(z)}{f(z)} - 1\right| \ (k \geq 0;\ z \in \mathbb{U}). \tag{8}$$

The classes of k-uniformly convex functions and k-starlike functions were introduced by Kanas and Wisniowska (see [7,8]).

Definition 1.4 [9, with $p = 1$]. For $-1 \leq A < B \leq 1$, $|\lambda| < \frac{\pi}{2}$ and $0 \leq \alpha < 1$, we say that a function $f(z) \in \mathcal{A}$ is in the class $R^\lambda(A, B, \alpha)$ if it satisfies the subordination condition:

$$e^{i\lambda} f'(z) \prec \cos\lambda\left[(1-\alpha)\frac{1+Az}{1+Bz} + \alpha\right] + i\sin\lambda. \tag{9}$$

Using the principle of subordination, $f(z) \in R^\lambda(A, B, \alpha)$ if and only if there exists function $w(z)$ satisfying $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that

$$e^{i\lambda} f'(z) = \cos\lambda\left[(1-\alpha)\frac{1+Aw(z)}{1+Bw(z)} + \alpha\right] + i\sin\lambda,$$

or, equivalently,

$$\left|\frac{e^{i\lambda}(f'(z)-1)}{Be^{i\lambda}f'(z) - [Be^{i\lambda} + (A-B)(1-\alpha)\cos\lambda]}\right| < 1 \ (z \in \mathbb{U}). \tag{10}$$

For suitable choices of A, B and α, we obtain the following subclasses:

(i) $R^\lambda(-1, 1, \alpha) = R^\lambda(\alpha)$ $(0 \leq \alpha < 1)$ (see Kanas and Srivastava [10]);

(ii) $R^\lambda(A, B, 0) = R^\lambda(A, B)$ $\left(-1 \leq A < B \leq 1,\ |\lambda| < \frac{\pi}{2}\right)$ (see Shukla and Dashrath [11]);

(iii) $R^0(-\beta, \beta, 0) = D(\beta)$ the class of functions $f(z) \in \mathcal{A}$ satisfying the condition

$$\left|\frac{f'(z)-1}{f'(z)+1}\right| < \beta \ (0 < \beta \leq 1;\ z \in \mathbb{U}),$$

introduced and studied by Padmanabhan [12] and later Caplinger and Causey [13];

(iv) $R^0(-\beta, \beta, \alpha) = R(\alpha, \beta)$ the class of functions $f(z) \in \mathcal{A}$ satisfying the condition

$$\left|\frac{f'(z)-1}{f'(z)+1-2\alpha}\right| < \beta \ (0 \leq \alpha < 1;\ 0 < \beta \leq 1;\ z \in \mathbb{U}),$$

studied by Junenja and Mogra [14].

Also, we note that:
$R^\lambda(-\beta, \beta, \alpha) = R^\lambda(\alpha, \beta) =$

$$\left\{f(z) \in \mathcal{A} : \left|\frac{f'(z)-1}{f'(z)-1+2(1-\alpha)e^{-i\lambda}\cos\lambda}\right| < \beta \ (|\lambda| < \frac{\pi}{2};\ 0 \leq \alpha < 1;\ 0 < \beta \leq 1;\ z \in \mathbb{U})\right\}.$$

The (Gaussian) hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \ (z \in \mathbb{U};\ c \neq 0, -1, -2, \ldots),$$

where

$$(\gamma)_n = \begin{cases} 1 & \text{if } n = 0, \\ \gamma(\gamma+1)(\gamma+2) \ldots (\gamma+n-1) & \text{if } n \in \mathbb{N} = \{1, 2, \ldots\}. \end{cases}$$

We note that ${}_2F_1(a, b; c; 1)$ converges for $\operatorname{Re}(c - a - b) > 0$ and is related to gamma function by

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{11}$$

Using the (Gaussian) hypergeometric function, consider the functions

$$g(a, b; c; z) = z\, {}_2F_1(a, b; c; z) \ (z \in \mathbb{U}), \tag{12}$$

$$h_\mu(a, b; c; z) = (1-\mu)[g(a, b; c; z)] + \mu z[g(a, b; c; z)]' \ (z \in \mathbb{U};\ \mu \geq 0), \tag{13}$$

and

$$J_{\mu,\delta}(a, b; c; z) = (1-\mu+\delta)[g(a, b; c; z)] + (\mu-\delta)z[g(a, b; c; z)]' + \mu\delta z^2[g(a, b; c; z)]'' \ (z \in \mathbb{U};\ \mu, \delta \geq 0;\ \mu \geq \delta). \tag{14}$$

The mapping properties of functions $h_\mu(a, b; c; z)$ and $J_{\mu,\delta}(a, b; c; z)$ were studied by Shukla and Shukla [15] and Tang and Deng [16, with $p = 1$], respectively.

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