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Review Paper

On the existence of solutions of two differential equations with a nonlocal condition

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Abstract In this paper we study the existence of solutions of two Cauchy problems of two nonlinear differential equations with nonlocal condition. The continuous dependence of the solutions on the coefficients of the nonlocal condition will be studied.

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1. Introduction

Problems with nonlocal conditions have been extensively studied by several authors in the last two decades. The reader is referred to [1–5] and [6–13] and references therein.

Consider the two nonlinear differential equations

$$\frac{dx(t)}{dt} = f\left(t, x(t), \frac{dx(t)}{dt}\right), \quad t \in (0, T], \tag{1}$$

and

$$\frac{dx(t)}{dt} = g\left(t, x(t), \frac{dx(t)}{dt}\right), \quad a.e.t \in (0, T], \tag{2}$$

with the nonlocal condition

$$\sum_{k=1}^{m} a_k x(\tau_k) = x_o, \quad \tau_k \in (0, T). \tag{3}$$

Our aim here is to study the existence of solutions for the two problems (1) with the nonlocal condition (3) and (2) with the nonlocal condition (3). Moreover, the continuous dependence of the solutions of the above two problems on x_0 and the nonlocal coefficients a_k will be studied.

2. Functional integral equations

Lemma 2.1. *Let* $\sum_{k=1}^{m} a_k \neq 0$. *The solution of the nonlocal problem (1) and (3) can be expressed by the integral equation*

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$$x(t) = A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds\right) + \int_0^t y(s)ds,$$
$$A = \left(\sum_{k=1}^{m} a_k\right)^{-1}, \quad (4)$$

where y is the solution of the functional integral equation

$$y(t) = f\left(t, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds + \int_0^t y(s)ds, y(t)\right),$$
$$t \in [0, T]. \quad (5)$$

Proof. Let $\frac{dx(t)}{dt} = y(t)$ in Eq. (1), then we obtain

$$y(t) = f(t, x(t), y(t))$$

where

$$x(t) = x(0) + \int_0^t y(s)ds. \quad (6)$$

Letting $t = \tau_k$in (6), we obtain

$$\sum_{k=1}^{m} a_k x(\tau_k) = \sum_{k=1}^{m} a_k x(0) + \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds. \quad (7)$$

Then

$$x(0) = A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds\right) \quad (8)$$

where $A = (\sum_{k=1}^{m} a_k)^{-1}$.

And we obtain

$$x(t) = A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds\right) + \int_0^t y(s)ds,$$

where y is the solution of the functional integral equation

$$y(t) = f\left(t, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds + \int_0^t y(s)ds, y(t)\right),$$
$$t \in [0, T].$$

By similar way, the following lemma can be proved. □

Lemma 2.2. *Let $\sum_{k=1}^{m} a_k \neq 0$. The solution of the nonlocal problem (2) and (3) can be expressed by the integral equation*

$$x(t) = A\left(x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds\right) + \int_0^t y(s)ds,$$
$$A = \left(\sum_{k=1}^{m} a_k\right)^{-1}, \quad (9)$$

where y is the solution of the functional integral equation

$$y(t) = g\left(t, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)ds + \int_0^t y(s)ds, y(t)\right),$$
$$t \in [0, T]. \quad (10)$$

2.1. *Existence of solutions*

Consider the functional integral equations (5) and (10) with the following assumptions:

(i) f: $[0, T] \times R \times R \to R$ is continuous and satisfies Lipschitz condition

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \le M_1(|u_1 - v_1| + |u_2 - v_2|),$$

(ii) g: $[0, T] \times R \times R \to R$ is measurable in $t \in [0, T]$ for any $(u_1, u_2) \in R \times R$ and satisfies Lipschitz condition

$$|g(t, u_1, u_2) - g(t, v_1, v_2)| \le M_2(|u_1 - v_1| + |u_2 - v_2|),$$

and

$$\int_0^t |g(t, 0, 0)|dt \le N$$

(iii) $M^* = M_1(2T + 1) < 1$
(iv) $M^{**} = M_2(2T + 1) < 1$.

Now we have the following theorem

Theorem 2.1. *Let the assumptions (i) and (iii) be satisfied. Then the functional integral equation (5) has a unique solution $y \in C[0, T]$.*

Proof. Define the operator H by

$$Hy(t) = f\left(t, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^t y(s)\,ds,\ y(t)\right),$$
$$t \in [0, T]. \quad (11)$$

Let $y \in C[0, T]$, then

$$\begin{aligned}
&|Hy(t_2) - Hy(t_1)| \\
&= \left| f\left(t_2, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^{t_2} y(s)\,ds,\ y(t_2)\right)\right. \\
&\quad \left. - f\left(t_2, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^{t_2} y(s)\,ds,\ y(t_1)\right)\right| \\
&\quad + \left| f\left(t_2, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^{t_2} y(s)\,ds,\ y(t_1)\right)\right. \\
&\quad \left. - f\left(t_2, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^{t_1} y(s)\,ds,\ y(t_1)\right)\right| \\
&\quad + \left| f\left(t_2, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^{t_1} y(s)\,ds,\ y(t_1)\right)\right. \\
&\quad \left. - f\left(t_1, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^{t_1} y(s)\,ds,\ y(t_1)\right)\right| \\
&\le M_1 \int_{t_1}^{t_2} |y(s)|ds + M_1|y(t_2) - y(t_1)| \\
&\quad + \left| f\left(t_2, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^{t_1} y(s)\,ds,\ y(t_1)\right)\right. \\
&\quad \left. - f\left(t_1, Ax_0 - A\sum_{k=1}^{m} a_k \int_0^{\tau_k} y(s)\,ds + \int_0^{t_1} y(s)\,ds, y(t_1)\right)\right|
\end{aligned}$$