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Original Article

A spectral Rayleigh–Ritz scheme for nonlinear partial differential systems of first order

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Abstract Nonlinear systems of partial differential problems of first order with Dirichlet boundary conditions is considered. ultraspherical integral zero- boundary (UIZB) method is combined with Rayleigh–Ritz method to approximate the unknowns. The approach converts the problem to be a multi-objective constrained optimization problem which is easier to solve. Accurate results can be obtained by selecting a limited number of collocation points. Numerical examples are included to demonstrate the accuracy and efficiency of the propped method.

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1. Introduction

For last few decades, spectral methods using expansion in orthogonal polynomials such as Chebyshev or ultraspherical polynomials (see for instance [1,2]) is well-known for its high accuracy. The pseudospectral method has been developed to obtain more accurate solutions in scientific computation. Doha et al. [3] constructed the Jacobi–Gauss–Lobatto pseudospectral schemes for numerically solving a certain nonlinear Schrodinger equations. Doha et al. [4] investigated a Chebyshev–Gauss–Radau collocation method in combination with the implicit Runge–Kutta scheme to obtain more accurate numerical solutions for hyperbolic systems of first order. Naher et al. [5] proposed extension of the generalized and improved (G'/G)-expansion method for constructing class of exact traveling wave solutions of nonlinear evolution equations. Demiray et al. [6] combine the $(G0/G;\ 1/G)$-expansion method with Maple to obtain exact travelling wave solutions of the nonlinear wave equations.

Rayleigh–Ritz method is used to convert differential equations to a minimization problem for certain criteria. Many papers discussed the use of this method to solve several problems, such as modeling the expansion of an elastic body [7], approximating part of the spectrum of an elliptic operator [8] and

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obtaining results for the time period and deflection of certain modes of vibration of rectangular plates [9].

According to the close relation of boundary value problems of partial differential equations to physical applications, the theory of boundary value problems is very rich. Several technological processes and scientific applications yield boundary value problems for PDE's. Howison and Oliver [10] analyzed a free boundary problem arising in a model for inviscid, incompressible shallow water entry at small deadrise angles. El Dhaba et al. [11] used a boundary integral method to solve a problem of uncoupled magnetothermoelasticity for an infinite, elliptical cylindrical conductor carrying a steady axial and uniform electric current. Khanday [12] treated the temperature distribution in multi-layered human skin and subcutaneous tissues and suggested a model of the solution of parabolic heat equation.

Chen [13] studied a free boundary value problem of the Euler system arising in the inviscid steady supersonic flow past a symmetric curved cone. Tsai [14] combined the homotopy analysis method with the method of fundamental solutions and the augmented polyharmonic spline to solve certain nonlinear partial differential equations. Feng et al. [15] developed a new framework for designing and analyzing convergent finite difference methods for approximating both classical and viscosity solutions of second order fully nonlinear partial differential equations (PDEs) in 1-D. Hosseini et al. [16] applied the operational Tau method with arbitrary polynomial bases to approximate the solution of a class of nonlinear transient heat conduction equations with some supplementary conditions.

Khalil et al. [17] developed an operational matrix with shifted Legendre polynomials to approximate solution of fractional differential equations(FDEs) and coupled system of FDEs with variable coefficients. The proposed method converts the problem into a system of easily solvable algebraic equations. The authors discussed also the convergence of the scheme and solved some test problems to show the efficiency and applicability of the method. Khalil et al. [18] extended the idea of pseudo spectral method to approximate solution of time fractional order three-dimensional heat conduction equations on a cubic domain. They studied shifted Jacobi polynomials and provide a simple scheme to approximate function of multi variables in terms of these polynomials. They developed operational matrices for arbitrary order integrations as well as for arbitrary order derivatives.

In the present paper, we numerically solve partial differential systems of first order. In fact, we treat with this problem as follows: We use ultraspherical integral method zero- boundary (UIZB) method to approximate the unknowns. We apply Rayleigh–Ritz method to reformulate the problem to be multi-objective constrained optimization problem. The resulting constrained optimization problem is then solved by sequential minimization processes of the Penalty leap frog method.

The outline of this paper is arranged as follows. In the next section, some properties of ultraspherical polynomials and ultraspherical integral matrix is investigated. In Section 3 Model of the problem is introduced. In Section 4, the proposed method, namely, the ultraspherical integral zero- boundary-Rayleigh–Ritz (UIZB-RR) method is constructed for solving the proposed problem. Error estimates and convergence index is investigated in Section 5. Some numerical examples are proposed in Section 6 to show the accuracy of our method. Finally, in Section 7, some observations and conclusions are presented.

2. Ultraspherical integral method

The ultraspherical polynomials $\{G_k(\lambda,\ x)\}_{k=0}^{\infty}$, where $\lambda > -0.5$ is a parameter, are defined by:

$$G_{k+1}(\lambda, x) = \frac{2(k+\lambda)}{k+2\lambda} x G_k(\lambda, x) - \frac{k}{k+2\lambda} G_{k-1}(\lambda, x), \quad k = 1, 2, \ldots, \tag{2.1}$$

$$G_k(\lambda, x) = \frac{d}{dx}\left[\frac{1}{2(k+1)} G_{k+1}(\lambda, x) - \frac{k}{2(k+2\lambda)(k+2\lambda-1)} G_{k-1}(\lambda, x)\right]. \tag{2.2}$$

Eq. (2.1) defines the ultraspherical polynomials starting with $G_1(\lambda,\ x) = x,\ G_0(\lambda,\ x) = 1$, whereas Eq. (2.2) can be used to define the integration of the ultraspherical polynomials(by simple integration) see El-Hawary et al. [19].

We define the collocation points to be the ultraspherical zeros points combined with the two boundary points of the interval, that is:

$$\Lambda = \{x_j |\ G_N(\lambda, x_j) = 0, k_j = 1, 2, \ldots, N-1, x_0 = -1, x_N = 1\}. \tag{2.3}$$

With this definition, we have

$$\int_{-1}^{x_i} f(x)dx = \sum_{k_j=0}^{N} S_{ij}^{[\lambda]} f(x_j), \tag{2.4}$$

where the element of the ultraspherical integral matrix of first degree S, are defined by [17]:

$$S_{ij}^{[\lambda]} = \sum_{k=0}^{N} \frac{\varpi_j G_k(\lambda, x_j)}{\alpha_k} \int_{-1}^{x_i} G_k(x)dx, \quad i,\ j, = 0, \quad 1, \ldots N, \tag{2.5}$$

and $G_k(x)$ is the ultraspherical polynomial of degree k, where ϖ_j and α_k obtained by

$$\varpi_j = \frac{1}{\sum_{k=0}^{N} \frac{\varpi_j (G_k(x_j))^2}{\lambda_k}},$$
$$\alpha_k = \frac{j!\Gamma(\lambda+.5)\Gamma(k+\lambda+.5)\Gamma(K+\lambda)\Gamma(2\lambda)}{2^{1-2k-2\lambda-\tau}\Gamma(2k+2\lambda+1)\Gamma(k+2\lambda)\Gamma(\lambda)}, \tag{2.6}$$

with

$$\tau = \begin{cases} 1, & if \quad \lambda = k = 0 \\ 0, & otherwise. \end{cases} \tag{2.7}$$

3. Model of the problem

We consider the general form of system of nonlinear boundary value problem of first order PDE. It can be defined by the following equations:

$$L_k\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, v, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) = \chi_k(x, y),$$
$$k = 1, 2 \quad and \quad (x, y) \in \varpi = [-1,\ 1] \times [-1,\ 1], \tag{3.1}$$

with Dirichlet boundary condition