Journal of the Egyptian Mathematical Society (2016) 24, 392–395

Egyptian Mathematical Society
Journal of the Egyptian Mathematical Society
www.etms-eg.org
www.elsevier.com/locate/joems

Original Article

Attractivity of the recursive sequence $x_{n+1} = (A - Bx_{n-2})/(C + Dx_{n-1})$

A.M. Ahmed [a], N.A. Eshtewy [b,*]

[a] Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City 11884, Cairo, Egypt
[b] Department of Mathematics, Faculty of Science Arish, Suez Canal University, 41522, Ismailia, Egypt

Received 13 June 2015; revised 31 July 2015; accepted 12 August 2015
Available online 21 October 2015

Keywords
Difference equation;
Attractivity;
Basin of attraction

Abstract In this paper, we investigate the global attractivity of the difference equation

$$x_{n+1} = \frac{A - Bx_{n-2}}{C + Dx_{n-1}}, n = 0, 1, \ldots,$$

where A, B are nonnegative real numbers, C, D are positive real numbers and $C + Dx_{n-1} \neq 0$ for all $n \geq 0$.

2010 Mathematics Subject Classification: 39A10; 39A11

1. Introduction

Difference equations appear naturally as discrete analogues and numerical solutions of differential equations and delay differential equations having applications in biology, ecology, physics, etc. The qualitative study of difference equations is a fertile research area and increasingly attracts many mathematicians. This topic draws its importance from the fact that many real life phenomena are modeled using difference equations. The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations.

R. Abo-Zeid [1] investigated the attractivity of two nonlinear third order difference equations

$$x_{n+1} = \frac{A - Bx_{n-1}}{\pm C + Dx_{n-2}}, \quad n = 0, 1, \ldots,$$

where A, B are nonnegative real numbers, C, D are positive real numbers and $C + Dx_{n-2} \neq 0$ for all $n \geq 0$.

El-Owaidy et al. [2] investigated the global attractivity of the difference equation

$$x_{n+1} = \frac{\alpha - \beta x_{n-1}}{\gamma + x_n}, \quad n = 0, 1, \ldots,$$

where α, β, γ are non-negative real numbers and $\gamma + x_n \neq 0$ for all $n \geq 0$.

* Corresponding author. Tel.: +20 1067179166.
E-mail addresses: ahmedelkb@yahoo.com (A.M. Ahmed), neveena@ymail.com (N.A. Eshtewy).
Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

S1110-256X(15)00068-1
http://dx.doi.org/10.1016/j.joems.2015.08.004

M. A. El-Moneam [3] studied the global behavior of the higher order nonlinear rational difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + Dx_{n-\sigma} + \frac{bx_{n-k}}{|dx_{n-k} - ex_{n-l}|},\quad n = 0, 1, \ldots,$$

where the coefficients $A, B, C, D, b, d, e \in (0, \infty)$, while k, l and σ are positive integers. The initial conditions $x_{-\sigma}, \ldots, x_{-l}, \ldots, x_{-k}, \ldots, x_{-1}, x_0$ are arbitrary positive real numbers such that $k < l < \sigma$.

A. E. Hamza et al. [4] investigated the global asymptotic stability of the recursive sequence

$$x_{n+1} = \frac{\alpha - \beta x_n}{\gamma + x_{n-1}}, \qquad n = 0, 1, \ldots,$$

where $\alpha, \beta, \gamma \geq 0$. For other related results, see [5,6]. In this paper we study the global attractivity of the difference equations

$$x_{n+1} = \frac{A - Bx_{n-2}}{C + Dx_{n-1}}, \qquad n = 0, 1, \ldots, \tag{1.1}$$

where A, B are non negative real numbers, C, D are positive real numbers and $C + Dx_{n-1} \neq 0$ for all $n \geq 0$.

Theorem 1.1 ([6]). *Consider the third-degree polynomial equation*

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \tag{1.2}$$

where a_1, a_0 and a_2 are real numbers. Then a necessary and sufficient condition for all roots of Eq. (1.2) to lie inside the open disk $|\lambda| < 1$ is

$$|a_2 + a_0| < 1 + a_1, \quad |a_2 - 3a_0| < 3 - a_1 \text{ and } a_0^2 + a_1 - a_0a_2 < 1.$$

The change of variables $x_n = \frac{C}{D}y_n$ reduces Eq. (1.1) to the difference equation

$$y_{n+1} = \frac{p - qy_{n-2}}{1 + y_{n-1}}, \qquad n = 0, 1, \ldots. \tag{1.3}$$

where $p = \frac{AD}{C^2}$ and $q = \frac{B}{C}$.

2. The recursive sequence $y_{n+1} = (p - qy_{n-2})/(1 + y_{n-1})$

In this section we study the global attractivity of the difference equation

$$y_{n+1} = \frac{p - qy_{n-2}}{1 + y_{n-1}}, \qquad n = 0, 1, \ldots, \tag{2.1}$$

where p and q are positive real numbers.

The equilibrium points of Eq. (2.1) are the zeros of the function

$$f(\bar{y}) = \bar{y}^2 + (1 + q)\bar{y} - p.$$

That is

$$\bar{y}_1 = \frac{1}{2}(-(1 + q) + \sqrt{(1 + q)^2 + 4p})$$

and

$$\bar{y}_2 = \frac{1}{2}(-(1 + q) - \sqrt{(1 + q)^2 + 4p}).$$

The linearized equation associated with Eq. (2.1) about $\bar{y}_i$ is

$$z_{n+1} + \frac{\bar{y}_i}{1 + \bar{y}_i}z_{n-1} + \frac{q}{1 + \bar{y}_i}z_{n-2} = 0, \quad n = 0, 1, 2, \ldots.$$

Its associated characteristic equation is

$$\lambda^3 + \frac{\bar{y}_i}{1 + \bar{y}_i}\lambda + \frac{q}{1 + \bar{y}_i} = 0.$$

Suppose that

$$g_i(\lambda) = \lambda^3 + \frac{\bar{y}_i}{1 + \bar{y}_i}\lambda + \frac{q}{1 + \bar{y}_i}, \quad i = 1, 2. \tag{2.2}$$

Theorem 2.1.

(1) *The sufficient condition for the equilibrium point $\bar{y}_1$ to be locally asymptotically stable is $q \leq 1$.*
(2) *If $q > \frac{1}{3} + \sqrt{\frac{4}{3}p + \frac{4}{9}}$, then $\bar{y}_1$ is unstable.*
(3) *$\bar{y}_2$ is saddle equilibrium point.*

Proof.

(1) If $q \leq 1$, then by using Theorem 1.1 with $a_0 = \frac{q}{1+\bar{y}_i}$, $a_1 = \frac{\bar{y}_i}{1+\bar{y}_i}$, $a_2 = 0$. We can easily show that $\bar{y}_1$ is locally asymptotically stable.
(2) If $q > \frac{1}{3} + \sqrt{\frac{4}{3}p + \frac{4}{9}}$, then $g_1(\lambda)$ has a zero λ_1 in $(-\infty, -1)$, which implies that the equilibrium point $\bar{y}_1$ is unstable.
(3) It is clear that $g_2(\lambda)$ has a zero $\lambda_1 \in (0, 1)$, and $g_2(-\frac{q}{1+\bar{y}_2}) = \frac{q}{(1+\bar{y}_2)^3}[\bar{y}_2 + 1 - q^2]$.

It is clear that $g_2(\lambda)$ is an increasing function. Since $g_2(-\frac{q}{1+\bar{y}_2}) > 0$, then $\lambda_1 < -\frac{q}{1+\bar{y}_2} \Longrightarrow |\lambda_2\lambda_3| > 1 \Longrightarrow |\lambda_2| = |\lambda_3| > 1$, which implies that $\bar{y}_2$ is unstable equilibrium point (saddle). □

Lemma 1. *Assume that $q \leq 1$. Then the interval $[0, \frac{p}{q}]$ is an invariant interval for Eq. (2.1).*

Proof. Let $\{y_n\}_{n=-2}^{\infty}$ be a solution of Eq. (2.1) with $y_{-2}, y_{-1}, y_0 \in [0, \frac{p}{q}]$.

Consider the function $U_1(x, y) = \frac{p-qy}{1+x}$, U_1 is decreasing in x and y on $(-1, \infty) \times (-\infty, \frac{p}{q})$.

Hence

$0 = U_1(\frac{p}{q}, \frac{p}{q}) \leq y_1 = U_1(y_{-1}, y_{-2}) < U_1(0, 0) = p < \frac{p}{q}$,

by induction we obtain $0 \leq y_n \leq \frac{p}{q}\ \forall n \geq 1$.

Assume that there exists $k \geq 2$ such that the following conditions hold

$p \geq kq^2$ and $1 \geq \frac{kp}{q}$. □

Lemma 2. *Assume that condition (2.3) hold for some $k \geq 2$. Let $\{y_n\}$ be a solution of Eq. (2.1)*

If $y_n, y_{n+1}, y_{n+2} \in [-(k-1)\frac{p}{q}, \frac{p}{q}]$ for some $n \geq -2$, then $y_{n+i} \in [0, \frac{p}{q}]\ \forall i \geq 3$.