Journal of the Egyptian Mathematical Society (2016) **24**, 402–409

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

Original Article

Fixed point theorems in complex valued metric spaces

CrossMark

Naval Singh [a], Deepak Singh [b,*], Anil Badal [c], Vishal Joshi [d]

[a] *Govt. Science and Commerce College, Benazeer, Bhopal, Madhya Pradesh, India*
[b] *Department of Applied Sciences, NITTTR, Ministry of HRD, Govt. of India, Bhopal, Madhya Pradesh 462002, India*
[c] *Department of Mathematics, Barkatullah University, Bhopal, Madhya Pradesh, India*
[d] *Department of Mathematics, Jabalpur Engineering College, Jabalpur, Madhya Pradesh, India*

Received 16 May 2014; revised 9 February 2015; accepted 25 April 2015
Available online 9 September 2015

Keywords

Common fixed point;
Complex valued metric spaces;
Complete complex valued metric spaces;
Cauchy sequence;
Fixed point

Abstract The aim of this paper is to establish and prove several results on common fixed point for a pair of mappings satisfying more general contraction conditions portrayed by rational expressions having point-dependent control functions as coefficients in complex valued metric spaces. Our results generalize and extend the results of Azam et al. (2011) [1], Sintunavarat and Kumam (2012) [2], Rouzkard and Imdad (2012) [3], Sitthikul and Saejung (2012) [4] and Dass and Gupta (1975) [5]. To substantiate the authenticity of our results and to distinguish them from existing ones, some illustrative examples are also furnished.

2010 Mathematics Subject Classification: 47H10; 54H25; 30L99

1. Introduction

In 2011, Azam et al. [1] introduced the notion of complex valued metric spaces and established some fixed point results for a pair of mappings for contraction condition satisfying a rational expression. This idea is intended to define rational expressions which are not meaningful in cone metric spaces and thus many such results of analysis cannot be generalized to cone metric spaces but to complex valued metric spaces.

Complex valued metric space is useful in many branches of Mathematics, including algebraic geometry, number theory,

* Corresponding author. Tel.: +919826991961.
E-mail addresses: drsinghnaval12@gmail.com (N. Singh), dk.singh1002@gmail.com (D. Singh), badalanil80@rediffmail.com (A. Badal), joshivishal76@yahoo.in (V. Joshi).
Peer review under responsibility of Egyptian Mathematical Society.

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.joems.2015.04.005

applied Mathematics as well as in physics including hydrodynamics, mechanical engineering, thermodynamics and electrical engineering. After the establishment of complex valued metric spaces, Rouzkard et al. [3] established some common fixed point theorems satisfying certain rational expressions in these spaces to generalize the result of [1]. Subsequently Sintunavarat et al. [2,6] obtained common fixed point results by replacing the constant of contractive condition to control functions. Recently, Sitthikul et al. [4] established some fixed point results by generalizing the contractive conditions in the context of complex valued metric spaces. Many researchers have contributed with different concepts in this space. One can see in [7–13].

In what follows, we recall some notations and definitions due to Azam et al. [1], that will be used in our subsequent discussion.

Let C be the set of complex numbers and $z_1, z_2 \in C$. Define a partial order $\precsim$ on C as follows: $z_1 \precsim z_2$ if and only if $Re(z_1) \le Re(z_2)$ and $Im(z_1) \le Im(z_2)$. It follows that $z_1 \precsim z_2$ if one of the followings conditions is satisfied:

(C1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(C2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$,
(C3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$,
(C4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C2), (C3) and (C4) is satisfied and we will write $z_1 \prec z_2$ if only (C4) is satisfied.

Definition 1.1 ([1]). Let X be a non-empty set. A mapping d: $X \times X \to C$ is called a complex valued metric on X if the following conditions are satisfied:

(CM1) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$;
(CM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(CM3) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.

In this case, we say that (X, d) is a complex valued metric space.

Example 1.1. Let $X = C$ be a set of complex number. Define d: $C \times C \to C$. By

$$d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|,$$

where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (C, d) is a complex valued metric space.

Example 1.2 (inspired by [2]). Let $X = C$. Define a mapping d: $X \times X \to C$ by $d(z_1, z_2) = e^{ik}|z_1 - z_2|$, where $k \in [0, \frac{\pi}{2}]$. Then (X, d) is a complex valued metric space.

Definition 1.2. [1] Suppose that (X, d) is a complex valued metric space.

1. We say that a sequence $\{x_n\}$ is a *Cauchy sequence* if for every $0 \prec c \in C$ there exists an integer N such that $d(x_n, x_m) \prec c$ for all $n, m \ge N$.
2. We say that $\{x_n\}$ converges to an element $x \in X$ if for every $0 \prec c \in C$ there exists an integer N such that $d(x_n, x) \prec c$ for all $n \ge N$. In this case, we write $x_n \xrightarrow{d} x$.
3. We say that (X, d) is complete if every Cauchy sequence in X converge to a point in X.

Lemma 1.1. *[1] Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 1.2. *[1] Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

2. Main result

We start to this section with the following observation.

Proposition 2.1. *Let (X, d) be a complex valued metric space and S, T: $X \to X$. Let $x_0 \in X$ and defined the sequence $\{x_n\}$ b;y*

$$x_{2n+1} = Sx_{2n},$$
$$x_{2n+2} = Tx_{2n+1},\ \forall\, n = 0, 1, 2, \ldots \tag{2.1}$$

Assume that there exists a mapping λ: $X \times X \times X \to [0, 1)$ such that $\lambda(TSx, y, a) \le \lambda(x, y, a)$ and $\lambda(x, STy, a) \le \lambda(x, y, a), \forall x, y \in X$ and for a fixed element $a \in X$ and $n = 0, 1, 2, \ldots$. Then

$$\lambda(x_{2n}, y, a) \le \lambda(x_0, y, a) \quad \text{and} \quad \lambda(x, x_{2n+1}, a) \le \lambda(x, x_1, a).$$

Proof. Let $x, y \in X$ and $n = 0, 1, 2, \ldots$. Then we have

$$\lambda(x_{2n}, y, a) = \lambda(TSx_{2n-2}, y, a) \le \lambda(x_{2n-2}, y, a)$$
$$= \lambda(TSx_{2n-4}, y, a) \le \cdots \le \lambda(x_0, y, a).$$

Similarly, we have

$$\lambda(x, x_{2n+1}, a) = \lambda(x, STx_{2n-1}, a) \le \lambda(x, x_{2n-1}, a)$$
$$= \lambda(x, STx_{2n-3}, a) \le \cdots \le \lambda(x, x_1, a). \quad \square$$

The subsequent example illustrates the preceding proposition.

Example 2.1. Let $X = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$. Define d: $X \times X \to C$ as $d(x, y) = i|x - y|$ then clearly (X, d) is a complex valued metric space also define self-mappings S and T by

$$S\Big(\frac{1}{n+1}\Big) = \frac{1}{n+2} = T\Big(\frac{1}{n+1}\Big), \ n = 0, 1, 2, 3, \ldots.$$

Choosing sequence $\{x_n\}$ as $x_n = \frac{1}{n+1}, n = 0, 1, 2, 3, \ldots$ Then $x_0 = 1 \in X$.

Clearly $Sx_{2n} = x_{2n+1}$ and $Tx_{2n+1} = x_{2n+2}$.

Consider a mapping λ: $X \times X \times X \to [0, 1)$ by $\lambda(x, y, a) = \frac{x}{6} + \frac{y}{8} + a$, for all $x, y \in X$ and for fixed $a = \frac{1}{2} \in X$, then $\lambda(x, y, a) = \frac{x}{6} + \frac{y}{8} + \frac{1}{2}$.

Undoubtedly

$$\lambda(TSx, y, a) \le \lambda(x, y, a) \text{ and } \lambda(x, STy, a) \le \lambda(x, y, a),$$

for all $x, y \in X$ and for fixed $a \in X$.

Consider

$$\lambda(x_{2n}, y, a) = \frac{1}{6(2n+1)} + \frac{y}{8} + \frac{1}{2} \ \le \ \frac{1}{6} + \frac{y}{8} + \frac{1}{2}$$
$$= \lambda(x_0, y, a),$$

that is $\lambda(x_{2n}, y, a) \le \lambda(x_0, y, a), n = 0, 1, 2 \ldots, \forall y \in X$ and for $a = \frac{1}{2} \in X$. Also consider