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Original Article

Bifurcation analysis of vertical transmission model with preventive strategy

Gosalamang Ricardo Kelatlhegile*, Moatlhodi Kgosimore

Department of Basic Sciences, Botswana College of Agriculture, Private Bag 0027, Gaborone, Botswana

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Abstract We formulate and analyze a deterministic mathematical model for the prevention of a disease transmitted horizontally and vertically in a population of varying size. The model incorporates prevention of disease on individuals at birth and adulthood and allows for natural recovery from infection. The main aim of the study is to investigate the impact of a preventive strategy applied at birth and at adulthood in reducing the disease burden. Bifurcation analysis is explored to determine existence conditions for establishment of the epidemic states. The results of the study showed that in addition to the disease-free equilibrium there exist multiple endemic equilibria for the model reproduction number below unity. These results may have serious implications on the design of intervention programs and public health policies. Numerical simulations were carried out to illustrate analytical results.

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1. Introduction

Mathematical models have become important tools in analyzing the spread and control of infectious diseases [1]. Understanding the transmission characteristics of infectious diseases in communities, regions and countries may lead to implementation of better approaches of mitigating against infections or epidemics. In particular, mathematical models are useful in building and testing theories, and in comparing, planning, implementing and evaluating various detection, prevention, therapeutic and control programs [1]. The results of such studies may contribute to formulation of appropriate public health policies and guide the design of other relevant studies and development of methods for data collection.

The phenomena of backward bifurcation usually expressed as a graph of the equilibrium infective population size I in terms of the basic reproduction number $\Re_0$ arise from backward

* Corresponding author. Tel.: +267 74116522; fax: +267 3928753.
E-mail address: gkelatlhegile@bca.bw, gkelatlhegile@gmail.com (G.R. Kelatlhegile).
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transfer of individuals back to the susceptible population or multiple groups and asymmetry between groups or multiple interaction mechanisms [2]. Studies of bifurcation analysis were explored in a number of epidemic models of which a few are cited herein, to characterize the behavior of equilibria, make predictions on the outcome of the epidemic and derive conditions for prevention and control of diseases [2–9]. These studies demonstrated existence of multiple positive endemic equilibria for values of the basic reproduction number below unity and a backward bifurcation when the reproduction number is one, particularly in models of vaccination [7,10–14] and/or vaccination and treatment [15].

Vertical transmission can be accomplished through transplacental transfer of disease agents (bacteria, viruses, parasites) from the mother to an embryo, fetus, or baby during pregnancy or childbirth [16]. In most studies, it is considered less important compared to horizontal transmission, because infectiousness and parturition usually occurs at different times [17]. A number of studies incorporating vertical transmission investigated the effects of various epidemiological and demographical factors on the disease transmission [2,14,17–19]. For instance, Busenbuerg and Cooke [18] constructed and analyzed various compartmental models with vertical transmission to obtain insight on the role of vertical transmission in disease epidemics. It is noted that vertical transmission alone may not cause an epidemic and there is a certain threshold above which it may or may not contribute the epidemic [19]. Thus, it is important to include vertical transmission in models to evaluate the extent to which it may contribute to the epidemic and appropriately inform design of public health policy.

Xue-Zhi et al. [8] constructed and analyzed an *SIS* model with limited resource for treatment. The study established occurrence of backward bifurcations due to insufficient capacity for treatment as well as existence of bistable endemic equilibria as a result of limited resources. Kribs-Zaleta and Velasco-Hernandez [7] analyzed a simple two dimensional SIS model with vaccination and found that the model exhibited backward bifurcation for some parameter values. Furthermore, the results indicated that a vaccination campaign meant to reduce the disease reproduction number below unity may fail to control the disease. van den Driessche and Watmough [9] found multiple stable equilibria which exhibited backward bifurcation and hysteresis for an *SIS* epidemic model with non-constant contact rate. Yicang and Hanwu [13] formulated an *SIS* model with pulse vaccination to study its dynamical behavior and established that the pulse vaccination was more effective than the proportional vaccination. Li and Ma [20] considered an *SIS* epidemic model with vaccination, temporary immunity, and varying total population size and derived three threshold parameters that govern the dynamics of the disease. Gao and Hethcote [17] considered an *SIS* model with density-dependent demographics which incorporated the effects of vertical transmission and derived contact and growth thresholds that characterized the transmission dynamics of the disease.

Most studies reviewed in this paper did not consider multiple group targeted interventions and also ignored effects of vertical transmission. In this paper, we modify the models in [2,3] to formulate and analyze a mathematical model appropriate for the implementation of prevention strategy at birth and susceptible stage for a disease transmitted horizontally and vertically. We assume that the disease upon recovery does not induce permanent immunity and recovered individuals become susceptible. Following [2], we employ elementary methods to establish bifurcation behavior based on the number of solutions of a quadratic equation. The major distinction between our work and those reviewed are: (i) the incorporation of vertical transmission and (ii) the implementation of preventive strategies targeting two groups of susceptibles (newly borns and susceptible adults).

The organization of this paper is as follows. Section 2 provides model formulation and analysis. In Section 3, we investigate the existence of equilibria of the model. In Section 4, we investigate the stability analysis of the model. In Section 5 some numerical simulations are displayed in detail and close with a discussion in Section 6.

2. Model formulation and analysis

2.1. Model formulation

We consider an epidemic model with preventive strategies (e.g. vaccination, educational campaigns) [10] that incorporates the effects of vertical transmission. We formulate a mathematical model consisting of four compartments of susceptibles ($S(t)$), vaccinated susceptibles ($V(t)$), (comprising individuals protected against infection), infectives ($I(t)$) (assumed to be infectious) and the recovered ($R(t)$). The total host population is given by $N(t) = S(t) + V(t) + I(t) + R(t)$. The susceptible class is replenished through recruitment or births of unvaccinated individuals at a constant rate $(1 - \theta\epsilon)\pi_s$ and through births or recruitment of infection-free individuals from infected individuals at a rate $b\omega(1 - \theta\epsilon)$. A proportion $\theta, (0 \le \theta \le 1)$ of new susceptible individuals are vaccinated and the vaccine produces a protective immunological response at rate ϵ, $(0 \le \epsilon \le 1)$ of those vaccinated. This process results in a fraction $1 - \theta\epsilon$ of new susceptibles entering the susceptible population, while the fraction $\theta\epsilon$ enters the vaccinated (protected) population. The susceptible individuals are vaccinated at a constant rate ψ and also enters the vaccinated compartment. The susceptible population acquires infection at the rate βIS, where β is the infectivity rate. A proportion ω $(0 \le \omega \le 1)$ of new births are born infected through mother-to-child transmission (MTCT) and the remaining $(1 - \omega)$ infection-free newly borns are subjected to vaccination at a rate $\theta\epsilon$. Since vaccination induces protection among those vaccinated, we assume that vaccinated individuals acquire infection at a discounted rate $\rho\beta I$, $(0 \le \rho \le 1)$, where $\rho = 0$ means vaccine is perfect and $\rho = 1$ means that the vaccine is useless. We assume vaccine-induced immunity decays exponentially at a constant rate σ. All the compartments are subjected to natural mortality at per capita rate μ. Infected individuals further experience excess mortality due to infection at a constant rate δ. Infected class recover with temporary immunity at a constant rate γ and join a class of the recovered. Individuals in the recovered class loose their immunity at a constant rate α and return to the pool of susceptibles. The above description leads to the following system of differential equations:

$$\begin{aligned}
\dot{S}(t) &= \pi_s(1 - \theta\epsilon) + b(1 - \omega)(1 - \theta\epsilon)I \\
&\quad -\beta IS - (\mu + \psi)S + \alpha R + \sigma V, \\
\dot{V}(t) &= \pi_s\theta\epsilon + \psi S + b(1 - \omega)\theta\epsilon I - \rho\beta IV - (\mu + \sigma)V, \\
\dot{I}(t) &= b\omega I + \beta IS + \rho\beta IV - (\mu + \gamma + \delta)I, \\
\dot{R}(t) &= \gamma I - (\mu + \alpha)R.
\end{aligned} \tag{1}$$