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Some relations between power graphs and Cayley graphs

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Abstract Motivated by an open problem of Abawajy et al. [1] we find some relations between power graphs and Cayley graphs of finite cyclic groups. We show that the vertex deleted subgraphs of some power graphs are spanning subgraphs or equal to the complement of vertex deleted subgraphs of some unitary Cayley graphs. Also we prove that some Cayley graphs can be expressed as direct product of power graphs. Applying these relations we study the eigenvalues and energy of power graphs and the related Cayley graphs.

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1. Introduction

The study of graphical representation of semigroups or groups becomes an exciting research topic in the last few decades, leading to many fascinating results and questions. In this context the most popular class of graphs are the Cayley graphs. Cayley graphs are introduced in 1878, well studied and having many applications. The concept of power graphs is a very recent development. In this paper we have worked on an open problem of Abawajy et al. [1, Problem 10] which asked for link between power graphs and Cayley graphs.

The concept of directed power graph was first introduced and studied by Kelarev and Quinn [2–4]. The *directed power graph* of a semigroup S is a digraph with vertex set S and for $x, y \in S$ there is an arc from x to y if and only if $x \neq y$ and $y = x^m$ for some positive integer m. Following this Chakrabarty et al. [5] defined the *undirected power graph* $\mathcal{G}(G)$ of a group G as an undirected graph whose vertex set is G and two vertices u, v are adjacent if and only if $u \neq v$ and $u^m = v$ or $v^m = u$ for some positive integer m. After that the undirected power graph became the main focus of study in [6–9]. In [5] it was shown that for any finite group G, the power graph of a subgroup of G is an induced subgraph of $\mathcal{G}(G)$ and $\mathcal{G}(G)$ is complete if and only if G is a cyclic group of order 1 or p^m, for some prime p and positive integer m. In [6] Cameron has proved that for a finite cyclic group G of non-prime-power order n, the set of vertices T_n of $\mathcal{G}(G)$ which are adjacent to

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all other vertices of $\mathcal{G}(G)$, consists of the identity and the generators of G, so that $|T_n| = 1 + \phi(n)$, where $\phi(n)$ is the Euler's ϕ function. For more results on power graphs we refer the recent survey paper [1]. In this paper our main subject of study is undirected power graph and so we use the brief term '*power graph*' to refer to the *undirected power graph.*

For a finite group G and a subset S of G not containing the identity element e and satisfying $S^{-1} = \{s^{-1} : s \in S\} = S$, the *Cayley graph* of G with connection set S, $Cay(G, S)$, is an undirected graph with vertex set G and two vertices g and h are adjacent if and only if $gh^{-1} \in S$. For any positive integer n, let $\mathbb{Z}_n$ denotes the additive cyclic group of integers modulo n. If we represent the elements of $\mathbb{Z}_n$ by $\overline{0}, \overline{1}, \ldots, \overline{n-1}$, then $U_n = \{\overline{a} \in \mathbb{Z}_n : gcd(a, n) = 1\}$ is a subset of $\mathbb{Z}_n$ of order $\phi(n)$, where $\phi(n)$ is the Euler's ϕ function. The Cayley graph $Cay(\mathbb{Z}_n, U_n)$ is known as *unitary Cayley graph*, see [10]. One can observe that $Cay(\mathbb{Z}_n, \mathbb{Z}_n - \{\overline{0}\})$ is the complete graph K_n on n vertices.

For a finite simple graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$, the adjacency matrix $A(G) = (a_{ij})$ is defined as an $n \times n$ matrix, where $a_{ij} = 1$ if $v_i \sim v_j$, and $a_{ij} = 0$ otherwise. The eigenvalues of $A(G)$ are also called the eigenvalues of G and denoted by $\lambda_i(G), i = 1, 2, \ldots, n$. Since $A(G)$ is a symmetric matrix, $\lambda_i(G)$'s are all real and so they can be ordered as $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$. By Perron Frobenious theorem, see [11], $\lambda_1(G)$ is always positive and $\lambda_1(G) > |\lambda_i(G)|$ for all $i = 2, 3, \ldots, n$. For the graph G, the *energy* $E(G)$ of G, introduced by Gutman [12], is the sum of the absolute values of all its eigenvalues. The concept of energy of graphs arose in chemistry. The total π-electron energy of some conjugated carbon molecule, computed using H*ü*ckel theory, coincides with the energy of its "molecular" graph [13]. One can easily check that the eigenvalues of the complete graph K_n are $n-1$ and -1 with respective multiplicities 1 and $n-1$ and so $E(K_n) = 2(n-1)$. A graph G on n vertices is called *hyperenergetic* if $E(G) > 2(n-1)$ [13]. In [14] the authors found energy of all unitary Cayley graphs and determined conditions for which they are hyperenergetic.

Due to the applications of Cayley graphs in automata theory as explained in the monograph [15] and other versatile areas, the authors of [1] have given an open problem (Problem 10) to investigate the relations of power graphs and Cayley graphs. In this paper we find some relations between power graphs of finite cyclic groups $\mathbb{Z}_n$ and the Cayley graphs. Applying these relations we obtain the eigenvalues and energy of $\mathcal{G}(\mathbb{Z}_n)$ as well as of the related Cayley graphs and also find the relations between the energy of power graphs and Cayley graphs.

2. Relations between power graphs and Cayley graphs

It is known [10] that if $n = p$ is a prime number, then the unitary Cayley graph $Cay(\mathbb{Z}_n, U_n)$ is the complete graph K_n and if $n = p^\alpha$ is a prime-power then it is a complete p-partite graph. So we have the observations below.

(i) For any prime $p, \mathcal{G}(\mathbb{Z}_p) = K_p = Cay(\mathbb{Z}_p, U_p)$.
(ii) If $n = p^\alpha$ for some prime p and a positive integer $\alpha > 1$ then $Cay(\mathbb{Z}_n, U_n)$ is a regular spanning subgraph of $\mathcal{G}(\mathbb{Z}_n)$.

Notations: Let T_n be a subset of $\mathbb{Z}_n$ consists of the identity and generators i.e. $T_n = U_n \cup \{\overline{0}\}$. We denote the vertex deleted subgraph $\mathcal{G}(\mathbb{Z}_n) - T_n$ of the graph $\mathcal{G}(\mathbb{Z}_n)$ by $\mathcal{G}^*(\mathbb{Z}_n)$ and similarly $Cay^*(\mathbb{Z}_n, U_n) = Cay(\mathbb{Z}_n, U_n) - T_n$. Again for any graph G let $\overline{G}$ be the complement of G.

Theorem 2.1 gives a relation between $\mathcal{G}^*(\mathbb{Z}_n)$ and $Cay^*(\mathbb{Z}_n, U_n)$ for some values of n. From the definition of power graph it is clear that the vertices of T_n are adjacent to all other vertices in $\mathcal{G}(\mathbb{Z}_n)$. So roughly speaking, this theorem gives an expression of $\mathcal{G}(\mathbb{Z}_n)$ in terms of $Cay(\mathbb{Z}_n, U_n)$ for a class of values of n. Now since $Cay(\mathbb{Z}_n, U_n)$ is highly symmetric and also widely studied in the literature, this theorem may help us to investigate the structure and various properties of $\mathcal{G}(\mathbb{Z}_n)$. For instance in the next section we apply this theorem to investigate the eigenvalues and energy of $\mathcal{G}(\mathbb{Z}_{pq})$ which may not be so easy to get otherwise.

Theorem 2.1. *If $n = p^\alpha q^\beta$, where p, q are distinct primes and α, β are positive integers, then $\mathcal{G}^*(\mathbb{Z}_n)$ is a spanning subgraph of $\overline{Cay^*(\mathbb{Z}_n, U_n)}$. These two graphs are equal if and only if $\alpha = 1 = \beta$.*

Proof. Both the graphs have the same vertex set $\mathbb{Z}_n - T_n$, where $T_n = U_n \cup \{\overline{0}\}$. Let $E_p = \{a\overline{p} \in \mathbb{Z}_n : q \nmid a\}, E_q = \{b\overline{q} \in \mathbb{Z}_n : p \nmid b\}$ and $E_{pq} = \{t\overline{pq} \in \mathbb{Z}_n\} - \{\overline{0}\}$. Then E_p, E_q, and E_{pq} are pairwise disjoint sets whose union is $\mathbb{Z}_n - T_n$.

First we look into the adjacency among the vertices in the graph $Cay^*(\mathbb{Z}_n, U_n)$. If possible suppose that for some integers c and $d, c\overline{p} \sim d\overline{p}$. Then there exists $\overline{s} \in U_n$ such that for some integer r,

$$c\overline{p} + \overline{s} = d\overline{p} \Rightarrow s = (d - c + rp^{\alpha-1}q^\beta)p$$

which is a contradiction because $p \nmid s$. So for any integers c and $d, c\overline{p} \nsim d\overline{p}$. Similarly it can be shown that for any integers c and $d, c\overline{q} \nsim d\overline{q}$. Thus no vertex of E_p, E_q is adjacent to a vertex in the same set and each vertex of E_{pq} is an isolated vertex in the graph $Cay^*(\mathbb{Z}_n, U_n)$. Now for any $\overline{s} \in U_n, \gcd(s, n) = 1$ and so there exist integers u and v such that $su + nv = 1$. Consider any vertex $a\overline{p}$ from E_p and $b\overline{q}$ from E_q. Then

$$ap - bq = (ap - bq)(su + nv) \equiv (ap - bq)su \quad (mod\ n).$$

Since $p \nmid apu - bqu$ as well as $q \nmid apu - bqu, (ap - bq)$ $u\overline{s} \in U_n$ and so $a\overline{p} \sim b\overline{q}$. Hence $Cay^*(\mathbb{Z}_n, U_n)$ is a complete bipartite graph with bipartition $E_p \cup E_q$ along with the isolated vertices E_{pq}. So in the graph $\overline{Cay^*(\mathbb{Z}_n, U_n)}$, none of the vertices of E_p is adjacent to any vertex of E_q and these are the only non-adjacency of vertices in $\overline{Cay^*(\mathbb{Z}_n, U_n)}$.

Next we check for the non-adjacency of vertices in the graph $\mathcal{G}^*(\mathbb{Z}_n)$. If possible suppose that $a\overline{p}$ of E_p is adjacent to $b\overline{q}$ of E_q in the graph $\mathcal{G}^*(\mathbb{Z}_n)$. Then for some positive integers m_1 and m_1',

$$a\overline{p} = m_1 b\overline{q} \quad \text{or} \quad b\overline{q} = m_1' a\overline{p}.$$

First consider $a\overline{p} = m_1 b\overline{q}$ which implies $ap = m_1 bq + m_2 p^\alpha q^\beta$ for some $m_2 \in \mathbb{Z}$. But then $q \mid ap$ which is a contradiction as q is a prime and $q \nmid p$ as well as $q \nmid a$. Hence $a\overline{p} \neq m_1 b\overline{q}$. Similarly it can be proved that $b\overline{q} \neq m_1' a\overline{p}$. Thus none of the vertices of E_p is adjacent to any vertex of E_q. Hence $\mathcal{G}^*(\mathbb{Z}_n)$ is a spanning subgraph of $\overline{Cay^*(\mathbb{Z}_n, U_n)}$.