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ORIGINAL ARTICLE

On some generalizations of certain retarded nonlinear integral inequalities with iterated integrals and an application in retarded differential equation

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Abstract In this paper, we investigate some new nonlinear retarded integral inequalities of Gronwall–Bellman–Pachpatte type. These inequalities generalize some former famous inequalities and can be used as handy tools to study the qualitative as well as the quantitative properties of solutions of some nonlinear retarded differential and integral equations. An application is also presented to illustrate the usefulness of some of our results in estimation of solution of certain retarded nonlinear differential equations with the initial conditions.

MATHEMATICS SUBJECT CLASSIFICATION: 26D10; 26D15; 26D20; 34A12; 34A40

1. Introduction

Integral inequalities involving functions of one independent variable, which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of linear and nonlinear ordinary differential equations, integral equations, and differential-integral equations, see for instance [1–3]. One of the best known and widely used inequalities in the study of nonlinear differential equations is Gronwall–Bellman inequality [4,5], which has become one of the very few classical and most influential results in the theory and applications of inequalities. Because of its fundamental importance, over the years, many generalizations and analogous results of Gronwall–Bellman inequality have been established, such as [6–28]. Gronwall–Bellman inequality can be stated as follows:

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Theorem 1.1. *Let $f(t)$ and $u(t)$ be a real-valued nonnegative continuous functions defined on $D_1 = [0, h]$, and let u_0 and h be positive constants for which the inequality*

$$u(t) \leqslant u_0 + \int_0^t f(s)u(s)ds, \quad \forall t \in D_1.$$

Then

$$u(t) \leqslant u_0 \exp\left(\int_0^t f(s)ds\right), \quad \forall t \in D_1.$$

However, in certain situations the bounds provided by the above-mentioned inequalities are not directly applicable, and it is desirable to find some new estimates which will be equally important in order to achieve a diversity of desired goals. In the present paper we establish explicit bounds on retarded Gronwall–Bellman and Pachpatte-like inequalities and extend certain results that were proved be El-Owaidy et al. in [13], which can be used to study the qualitative behavior of the solutions of certain classes of retarded differential equations. In our results, there are not only composite functions of unknown functions in iterated integrals on the right hand side of our inequalities, but also the composite functions of unknown function exist in every layer of the iterated integrals, also we illustrate an application of our results, which verifies that our results are handy tools to study the qualitative properties of nonlinear differential equations and integral equations.

Theorem 1.2 (*Lipovan [10]*). *Let $u, f \in \mathcal{C}([t_0, T_0], \mathbb{R}_+)$. Further, let $\alpha \in \mathcal{C}([t_0, T_0], [t_0, T_0])$ be a nondecreasing with $\alpha(t) \leqslant t$ on $[t_0, T_0]$, and let k be a nonnegative constant.*

Then the inequality

$$u(t) \leqslant k + \int_{\alpha(t_0)}^{\alpha(t)} f(s)u(s)ds, \quad t_0 < t < T_0,$$

implies that

$$u(t) \leqslant k \exp\left(\int_{\alpha(t_0)}^{\alpha(t)} f(s)ds\right), \quad t_0 < t < T_0.$$

Theorem 1.3 (*Agarwal [11]*). *Let $\phi \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ be an increasing function, $u, a, f \in \mathcal{C}([t_0, T_0], \mathbb{R}_+)$, $a(t)$ be an increasing function, and $\alpha(t) \in \mathcal{C}([t_0, T_0], [t_0, T_0])$ be a nondecreasing with $\alpha(t) \leqslant t$ on $[t_0, T_0]$ where $T_0 \in (0, \infty)$ is a constant . Then the inequality*

$$u(t) \leqslant a(t) + \int_{\alpha(0)}^{\alpha(t)} f(s)\phi(u(s))ds, \quad t_0 < t < T_0,$$

implies that

$$u(t) \leqslant W^{-1}\left(W(a(t))\int_{\alpha(0)}^{\alpha(t)} f(s)ds\right), \quad t_0 < t < T_0,$$

where

$$W(t) = \int_1^t \frac{dt}{\phi(t)} ds, \quad t > 0,$$

W^{-1} is the inverse function of W, and T^* is the large number such that

$$W(a(T^*))\int_{\alpha(0)}^{\alpha(T^*)} f(s)ds \leqslant \int_1^\infty \frac{dt}{\phi(t)} ds.$$

2. Main results

In this section, several new generalized retarded integral inequalities of Gronwall–Bellman type are introduced. Throughout this article, $\mathbb{R}$ denoted the set of real numbers, $I = [0, \infty)$ is the subset of $\mathbb{R}$,$'$ denotes the derivative. $\mathcal{C}(I, I)$ denotes the set of all continuous functions from I into I and $\mathcal{C}^1(I, I)$ denotes the set of all continuously differentiable functions from I into I.

Theorem 2.1. *Let $u(t), g(t), f(t) \in \mathcal{C}(I, I)$ be nonnegative functions. We suppose that $\varphi, \varphi', \alpha \in \mathcal{C}^1(I, I)$ are increasing functions, with $\varphi'(t) \leqslant k, \varphi > 0, \alpha(t) \leqslant t, \alpha(0) = 0$, for all $t \in I; k, u_0$ be positive constants, If the inequality*

$$u(t) \leqslant u_0 + \int_0^{\alpha(t)} f(s)\varphi(u(s))\left[\varphi(u(s)) + \int_0^s g(\lambda)\varphi(u(\lambda))d\lambda\right]ds, \quad \forall t \in I, \tag{2.1}$$

holds, for all $t \in I$. Then

$$u(t) \leqslant \Phi^{-1}\left(\Phi(u_0) + \int_0^{\alpha(t)} f(s)\beta(\alpha^{-1}(s))ds\right), \quad \forall t \in I, \tag{2.2}$$

where

$$\Phi(r) = \int_1^r \frac{dt}{\varphi(t)}, \quad r > 0, \tag{2.3}$$

and

$$\beta(t) = \exp\left(\int_0^{\alpha(t)} g(s)ds\right)\left((\varphi^{-1}(u_0)) - k\int_0^{\alpha(t)} f(s)\exp\left(\int_0^s g(\lambda)d\lambda\right)ds\right)^{-1}, \tag{2.4}$$

for all $t \in I$, such that $(\varphi^{-1}(u_0)) - k\int_0^{\alpha(t)} f(s)\exp\left(\int_0^s g(\lambda)d\lambda\right)ds > 0, \forall t \in I$.

Proof. Let $z(t)$ denotes the function on the right-hand side of (2.1), which is a nonnegative and nondecreasing function on I with $z(0) = u_0$. Then (2.1) is equivalent to

$$u(t) \leqslant z(t), u(\alpha(t)) \leqslant z(\alpha(t)) \leqslant z(t), \qquad \forall t \in I. \tag{2.5}$$

Differentiating $z(t)$, with respect to t, we get

$$\frac{dz}{dt} = \alpha'(t)f(\alpha(t))\varphi(u(\alpha(t)))[\varphi(u(\alpha(t))) + \int_0^{\alpha(t)} g(s)\varphi(u(s))ds], \forall t \in I.$$

Using (2.5), we get

$$\frac{dz}{dt} \leqslant \alpha'(t)f(\alpha(t))\varphi(z(\alpha(t)))y(t), \forall t \in I, \tag{2.6}$$

where $y(t) = \varphi(z(t)) + \int_0^{\alpha(t)} g(s)\varphi(z(s))ds, y(0) = \varphi(z(0)) = \varphi(u_0)$, $y(t)$ is a nonnegative and nondecreasing function on I. By the monotonicity φ, φ', z and $\alpha(t) \leqslant t$ we have $\varphi(z(t)) \leqslant y(t)$, $\varphi'(z(t)) \leqslant k$. Differentiating $y(t)$ with respect to t, and using (2.6), we have

$$\begin{aligned}\frac{dy}{dt} &\leqslant \varphi'(z(t))\alpha'(t)f(\alpha(t))y^2(t) + \alpha'(t)g(\alpha(t))\varphi(z(t)) \\ &\leqslant k\alpha'(t)f(\alpha(t))y^2(t) + \alpha'(t)g(\alpha(t))y(t), \quad \forall t \in I.\end{aligned} \tag{2.7}$$

But $y(t) > 0$, from (2.7) we get