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ORIGINAL ARTICLE

Necessity and sufficiency for hypergeometric functions to be in a subclass of analytic functions

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Abstract The purpose of this paper is to introduce necessary and sufficient condition of (Gaussian) hypergeometric functions to be in a subclass of uniformly starlike and uniformly convex functions. Operators related to hypergeometric functions are also considered. Some of our results correct previously known results.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, and let $\mathcal{S}$ be the subclass of all functions in $\mathcal{A}$, which are univalent. Let $g(z) \in \mathcal{A}$, be given by

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n, \tag{1.2}$$

then, the integral convolution of two power series $f(z)$ and $g(z)$ is given by (see [1]):

$$(f \circledast g)(z) = z + \sum_{n=2}^{\infty} \frac{a_n g_n}{n} z^n = (g \circledast f)(z). \tag{1.3}$$

Let $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the subclasses of starlike and convex functions of order α, respectively. We note that $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$, the subclasses of starlike and convex functions (see, for example, Srivastava and Owa [2]).

Goodman [3,4] introduced the classes $\mathcal{UCV}$ and $\mathcal{UST}$ of uniformly convex and uniformly starlike functions. Following Goodman, Rønning [5] (see also [6]) gave one variable analytic characterization for $\mathcal{UCV}$, that is, a function $f(z)$ of the form (1.1) is in the class $\mathcal{UCV}$ if and only if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right| (z \in \mathbb{U}). \tag{1.4}$$

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Goodman proved the classical Alexander's result $f(z)\in\mathcal{UCV}\Longleftrightarrow zf'(z)\in\mathcal{UST}$, does not hold. On later, Rønning [7] introduced the class $\mathcal{S}_p$ which consists of functions such that $f(z)\in\mathcal{UCV}\Longleftrightarrow zf'(z)\in\mathcal{S}_p$. Also in [5], Rønning generalized the classes $\mathcal{UCV}$ and $\mathcal{S}_p$ by introducing a parameter α in the following.

Definition 1 [5]. A function $f(z)$ of the form (1.1) is in the class $\mathcal{S}_p(\alpha)$, if it satisfies the following condition:

$$\Re\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}>\left|\frac{zf'(z)}{f(z)}-1\right|(-1\leqslant\alpha<1;\ z\in\mathbb{U}),\qquad(1.5)$$

and $f(z)\in\mathcal{UCV}(\alpha)$, the class of uniformly convex functions of order α if and only if $zf'(z)\in\mathcal{S}_p(\alpha)$.

Also in [8], Bharati et al. introduced the classes $\mathcal{UCV}(\alpha,\beta)$ and $\mathcal{S}_p(\alpha,\beta)$ as follows:

Definition 2 [8]. A function $f(z)$ of the form (1.1) is said to be in the class $\mathcal{S}_p(\alpha,\beta)$, if it satisfies the following condition:

$$\Re\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}>\beta\left|\frac{zf'(z)}{f(z)}-1\right|\ (-1\leqslant\alpha<1;\beta\geqslant 0;z\in\mathbb{U}),\qquad(1.6)$$

and $f(z)\in\mathcal{UCV}(\alpha,\beta)$ if and only if $zf'(z)\in\mathcal{S}_p(\alpha,\beta)$.

Denote by $\mathcal{T}$, the subclass of $\mathcal{S}$ consisting of functions of the form:

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n\ (a_n\geqslant 0).\qquad(1.7)$$

Denote also by $\mathcal{T}^*(\alpha)=\mathcal{S}^*(\alpha)\cap\mathcal{T}$, $\mathcal{C}(\alpha)=\mathcal{K}(\alpha)\cap\mathcal{T}$, the subclasses of starlike and convex functions of order α with negative coefficients, which were introduced and studied by Silverman (see [9]). Also let $\mathcal{UCT}(\alpha)=\mathcal{UCV}(\alpha)\cap T$, $\mathcal{S}_p\mathcal{T}(\alpha)=\mathcal{S}_p(\alpha)\cap\mathcal{T}$, $\mathcal{UCT}(\alpha,\beta)=\mathcal{UCV}(\alpha,\beta)\cap T$ and $\mathcal{S}_p\mathcal{T}(\alpha,\beta)=\mathcal{S}_p(\alpha,\beta)\cap\mathcal{T}$.

Let $\mathcal{S}_\gamma(f;\alpha,\beta)$ $(-1\leqslant\alpha<1,\ \beta\geqslant 0$ and $0\leqslant\gamma\leqslant 1)$ be the subclass of $\mathcal{S}$ consisting of functions of the form (1.1) and satisfying the analytic criterion:

$$\Re\left\{\frac{zf'(z)+\gamma z^2f''(z)}{(1-\gamma)f(z)+\gamma zf'(z)}-\alpha\right\}>\beta\left|\frac{zf'(z)+\gamma z^2f''(z)}{(1-\gamma)f(z)+\gamma zf'(z)}-1\right|\ (z\in\mathbb{U}).\qquad(1.8)$$

The class $\mathcal{S}_\gamma(f;\alpha,\beta)$ was introduced and studied by Aouf et al. [10, with $g(z)=\frac{z}{1-z}$]. Further, we define the class $\mathcal{TS}_\gamma(f;\alpha,\beta)$ by

$\mathcal{TS}_\gamma(f;\alpha,\beta)=\mathcal{S}_\gamma(f;\alpha,\beta)\cap\mathcal{T}$.

Let $F(a,b;c;z)$ be the (Gaussian) hypergeometric function defined by

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n,$$

where $c\neq 0,\ -1,\ -2,\ \ldots$ and

$$(\lambda)_n=\begin{cases}1 & \text{if n}=0,\\ \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+\text{n}-1) & \text{if n}\in\mathbb{N}=\{1,2,\ldots\}.\end{cases}$$

We note that $F(a,b;\ c;\ 1)$ converges for $\Re(c-a-b)>0$ and is related to Gamma functions by

$$F(a,b;\ c;\ 1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.\qquad(1.9)$$

Also, we define the functions

$$g(a,b;\ c;\ z)=zF(a,b;\ c;\ z),\qquad(1.10)$$

and

$$h_\mu(a,b;\ c;\ z)=(1-\mu)(g(a,b;c;z))+\mu z(g(a,b;\ c;\ z))'(\mu\geqslant 0).\qquad(1.11)$$

The mapping properties of a function $h_\mu(a,b;\ c;\ z)$ was studied by Shukla and Shukla [11].

Corresponding to the Gaussian hypergeometric function ${}_2F_1(a,b;\ c;\ z)$, we define the linear operator $\mathcal{M}_{a,b,c}:\mathcal{A}\to\mathcal{A}$ by the integral convolution

$$[\mathcal{M}_{a,b,c}(f)](z)=g(a,b;\ c;\ z)\circledast f(z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\frac{a_n}{n}z^n\ (c\neq 0,-1,-2,\ldots),\qquad(1.12)$$

and the linear operator $\mathcal{N}_\mu:\mathcal{A}\to\mathcal{A}$ by the integral convolution

$$[\mathcal{N}_\mu(f)](z)=h_\mu(a,b;\ c;\ z)\circledast f(z)=z+\sum_{n=2}^{\infty}[1+\mu(n-1)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\frac{a_n}{n}z^n\times(c\neq 0,-1,-2,\ldots).\qquad(1.13)$$

Merkes and Scott [12] and Ruscheweyh and Singh [13] used continued fractions to find sufficient conditions for $zF(a,b;\ c;\ z)$ to be in the class $\mathcal{S}^*(\alpha)$ $(0\leqslant\alpha<1)$ for various choices of the parameters a, b and c. Carlson and Shaffer [14] showed how some convolution results about the class $\mathcal{S}^*(\alpha)$ may be expressed in terms of a linear operator acting on hypergeometric functions. Recently, Silverman [15] gave a necessary and sufficient conditions for $zF(a,b;\ c;\ z)$ to be in the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$.

2. Main results

Unless otherwise mentioned, we assume throughout this paper that $-1\leqslant\alpha<1,\ \beta\geqslant 0$ and $0\leqslant\gamma\leqslant 1$.
To establish our results, we need the following lemmas due to Aouf et al. [10].

Lemma 2.1 [10, Theorem 1, with $g(z)=\frac{z}{1-z}$]. *A sufficient condition for $f(z)$ defined by (1.1) to be in the class $\mathcal{S}_\gamma(f;\ \alpha,\beta)$ is*

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)][1+\gamma(n-1)]|a_n|\leqslant 1-\alpha.\qquad(2.1)$$

Lemma 2.2 [10, Theorem 2, with $g(z)=\frac{z}{1-z}$]. *A necessary and sufficient condition for $f(z)$ defined by (1.7) to be in the class $\mathcal{TS}_\gamma(f;\ \alpha,\beta)$ is*

$$\sum_{n=2}^{\infty}[n(1+\beta)-(\alpha+\beta)][1+\gamma(n-1)]a_n\leqslant 1-\alpha.\qquad(2.2)$$

By using Lemmas 2.1 *and* 2.2*, we get the following results.*