Journal of the Egyptian Mathematical Society (2015) **23**, 507–512

Egyptian Mathematical Society

**Journal of the Egyptian Mathematical Society**

www.etms-eg.org
www.elsevier.com/locate/joems

**ORIGINAL ARTICLE**

# Properties of superposition operators acting between $\mathcal{B}_{\mu}^{*}$ and $Q_{K}^{*}$

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Received 3 June 2014; revised 16 October 2014; accepted 3 January 2015
Available online 25 March 2015

**KEYWORDS**

Superposition operators;
$\mathcal{B}_{\mu}^{*}$;
Lipschitz continuity;
Compactness

**Abstract** In this paper we introduce natural metrics in the hyperbolic Bloch and $Q_K$-type spaces with respect to which these spaces are complete. Moreover, Lipschitz continuous, bounded and compact superposition operators $S_{\phi}$ from the hyperbolic Bloch type space to the hyperbolic $Q_K$-type space are characterized by conditions depending only on the analytic symbol $\phi$.

**2010 MATHEMATICS SUBJECT CLASSIFICATION:** 46*E*15; 47*B*33; 47*B*38; 54*C*35



## 1. Introduction

In 1979, Yamashita [1] introduced originally the concept of systematically hyperbolic function classes. Subsequently, this concept has studied for hyperbolic Hardy, BMOA and Dirichlet-classes (see, e.g., [1,3–7]). In the last decades, Smith [8] studied inner functions in the hyperbolic little Bloch-class. The hyperbolic counter parts of the $Q_p$-spaces were studied by Li [9] and Li et al. [10].

On the other hand, Cámera and Giménez [11,12] studied the Bergman space $A^p$, the space of all $L^p$ functions (with respect to Lebesgue area measure) which is analytic in the unit disk. They showed that $S_{\phi}(A^p) \subset A^q$ if and only if $\phi$ is a polynomial of degree at most $p/q$ where $S_{\phi} : L^p(\mathbb{D}) \to L^q(\mathbb{D})$ is the superposition operator. Later, Buckley and Vukotic [13,14] introduced superposition operators from Besov spaces into Bergman spaces and univalent interpolation in Besov spaces. Also, in [15], Alvarez et al. characterized superposition operators between the Bloch space and Bergman spaces. Recently, Wen Xu [16] studied superposition operators on Bloch-type spaces.

Let $X$ and $Y$ be two metric spaces of analytic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} :| z |< 1\}$. Assume that $\phi$ denotes a complex-valued function in the plane $\mathbb{C}$. The superposition operator $S_{\phi}$ on $X$defined by

$$S_{\phi}(f) := \phi \circ f, \quad f \in X.$$

If $\phi \circ f \in Y$ for $f \in X$, we say that $\phi$ acts by superposition from $X$into $Y$. As in Wen Xu [16] if $X$ contains linear functions, $\phi$ must be an analytic function.

Let $H(\mathbb{D})$ be the class of analytic functions on $\mathbb{D}$. Also, $B(\mathbb{D})$ denotes the class of all analytic functions on $\mathbb{D}$ such that $|f(z)| < 1$ for all $z \in \mathbb{D}$. It is clear that $B(\mathbb{D}) \subset H(\mathbb{D})$.

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Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.joems.2015.01.003
1110-256X 

Hyperbolic derivative for analytic functions on the unit disk $\mathbb{D}$.

$f^*(z) = \frac{|f'(z)|}{1-|f(z)|^2}$ (cf. [17]).

The spaces of analytic functions, have been actively appearing in different areas of mathematical sciences such as dynamical systems, theory of semigroups, probability, mathematical physics and quantum mechanics (see [18–20] and others). Now, we list the following definitions.

**Definition 1.1** [2]. Let $f$ be an analytic function in $\mathbb{D}$ and $0 < \alpha < \infty$. The $\alpha$-Bloch space $\mathcal{B}^\alpha$ is defined by

$$\mathcal{B}^\alpha = \left\{ f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty \right\},$$

the little $\alpha$-Bloch space $\mathcal{B}_0^\alpha$ is given as follows

$$\mathcal{B}_0^\alpha = \left\{ f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}_0^\alpha} = \lim_{|z|\to 1^-} (1-|z|^2)^\alpha |f'(z)| = 0 \right\}.$$

The spaces $\mathcal{B}^1$ and $\mathcal{B}_0^1$ are called as the Bloch space, and little Bloch space and denoted by $\mathcal{B}$ and $\mathcal{B}_0$ respectively (see [21]).

A positive continuous function $\mu$ on $[0,1)$ is called normal if there are three constants $0 \leqslant \delta < 1$ and $0 < a < b$ such that.

i. $\frac{\mu(r)}{(1-r)^a}$ is decreasing on $[\delta, 1)$ and $\lim_{r\to 1} \frac{\mu(r)}{(1-r)^a} = 0$;
ii. $\frac{\mu(r)}{(1-r)^b}$ is increasing on $[\delta, 1)$ and $\lim_{r\to 1} \frac{\mu(r)}{(1-r)^b} = \infty$.

**Definition 1.2** [22]. A function $f \in H(\mathbb{D})$ such that

$$\|f\|_\mu := \sup_{z\in\mathbb{D}} \mu(|z|) f'(z) < \infty$$

is called a $\mu$-Bloch function. The space of all $\mu$-Bloch functions is denoted by $\mathcal{B}_\mu$.

It is readily seen that $\mathcal{B}_\mu$ is a Banach space with the norm $\|f\|_{\mathcal{B}_\mu} := |f(0)| + \|f\|_\mu$. Also, when $\mu(z) = 1-|z|^2$, the space $\mathcal{B}_\mu$ is just the Bloch space which is denoted by $\mathcal{B}$; while when $\mu(z) = (1-|z|^2)^\alpha$ with $\alpha > 0$, the space $\mathcal{B}_\mu$ becomes the $\alpha$-Bloch space which is denoted by $\mathcal{B}_\alpha$.

The hyperbolic $\mu$-Bloch space is defined as follows:

**Definition 1.3** [23]. The sets of $f \in B(\mathbb{D})$ for which

$$\mathcal{B}_\mu^* = \left\{ f : f \text{ analytic in } \mathbb{D} \text{ and } \sup_{z\in\mathbb{D}} \mu(|z|) f^*(z) < \infty \right\}.$$

The little hyperbolic Bloch space $\mathcal{B}_{\mu,0}^*$ is a subspace of $\mathcal{B}_\mu^*$ consisting of all $f \in \mathcal{B}_\mu^*$ such that

$$\lim_{|z|\to 1^-} \mu(|z|) f^*(z) = 0.$$

Following [23], the authors defined a natural metric on the hyperbolic $\mu$-Bloch space $\mathcal{B}_\mu^*$ in the following way:

$$d(f,g;\mathcal{B}_\mu^*) := d_{\mathcal{B}_\mu^*}(f,g) + \|f-g\|_{\mathcal{B}_\mu} + |f(0)-g(0)|,$$

where

$$d_{\mathcal{B}_\mu^*}(f,g) := \sup_{a\in\mathbb{D}} \left| \frac{f'(z)}{1-|f(z)|^2} - \frac{g'(z)}{1-|g(z)|^2} \right| \mu(|z|)$$

for $f,\ g \in \mathcal{B}_\mu^*$.

The following conditions have played crucial roles in the study of $Q_K$ spaces:

$$\int_0^1 \phi_K(s) \frac{ds}{s} < \infty. \tag{1}$$

$$\int_1^\infty \phi_K(s) \frac{ds}{s^2} < \infty. \tag{2}$$

**Lemma 1.1** [24]. *If $K$ satisfy the condition (2), then the function*

$$K_1(t) = t \int_t^\infty \frac{K(s)}{s^2} ds \quad (\text{where, } 0 < t < \infty),$$

*has the following properties:*

*(A) $K_1$ is nondecreasing on $(0,\infty)$.*
*(B) $K_1(t)/t$ is nondecreasing on $(0,\infty)$.*
*(C) $K_1(t) \geqslant K(t)$ for all $t \in (0,\infty)$.*
*(D) $K_1 \lesssim K$ on $(0,1]$.*
*If $K(t) = K(1)$ for $t \geqslant 1$, then we also have*
*(E) $K_1(t) = K_1(1) = K(1)$ for $t \geqslant 1$, so $K_1 \approx K$ on $(0,\infty)$.*

**Lemma 1.2** [24]. *If $K$ satisfy the condition (2), then we can find another non-negative weight function given by*

$$K_1(t) = t \int_t^\infty \frac{K(s)}{s^2} ds \quad (\text{where, } 0 < t < \infty),$$

*such that $Q_K = Q_{K_1}$ and that the new function $K_1$ has the following properties:*

*(A) $K_1$ is nondecreasing on $(0,\infty)$.*
*(B) $K_1(t)/t$ is nondecreasing on $(0,\infty)$.*
*(c) $K_1(t)$ satisfies condition (1).*
*(d) $K_1(2t) \approx K_1(t)$ on $(0,\infty)$.*
*(e) $K_1(t) \approx K(t)$ on $(0,1]$.*
*(f) $K_1$ is differentiable on $(0,\infty)$.*
*(g) $K_1$ is concave on $(0,\infty)$.*
*(h) $K_1(t) = K_1(1)$ for $t \geqslant 1$.*

**Definition 1.4** (*see [25]*). Let a function $K : [0,\infty) \to [0,\infty)$. The space $Q_K$ is defined by

$$Q_K = \left\{ f \in H(\mathbb{D}) : \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(g(z,a))\, dA(z) < \infty \right\}.$$

If

$$\lim_{|a|\to 1^-} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(g(z,a))\, dA(z) = 0,$$

then $f \in Q_{K,0}$. Clearly, if $K(t) = t^p$, then $Q_K = Q_p$.

Li et al. [10] defined the hyperbolic $Q_K$ type space $Q_K^*$ as follows.

**Definition 1.5.** Let $K : [0,\infty) \to [0,\infty)$. The hyperbolic space $Q_K^*$ consists of those functions $f \in B(\mathbb{D})$ for which

$$\|f\|_{Q_K^*}^2 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (f^*(z))^2 K(g(z,a))\, dA(z) < \infty.$$