Journal of the Egyptian Mathematical Society (2015) 23, 518–522

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

On Akbar-Zadeh's theorem on a Finsler space of constant curvature ☆

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Received 13 July 2013; revised 8 July 2014; accepted 13 July 2014
Available online 11 February 2015

KEYWORDS

Cartan connection;
Akbar-Zadeh's theorem;
Symmetric manifold;
S_3-Like manifold;
S_4-Like manifold

Abstract The aim of the present paper is to give two *intrinsic* generalizations of Akbar-Zadeh's theorem on a Finsler space of constant curvature. Some consequences, of these generalizations, are drown.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 53C60; 53B40; 58B20; 53C12

0. Introduction

Using the tools of the traditional tensor calculus, in his paper [1], Akbar-Zadeh proved that if the h-curvature R^r_{ijk} of the Cartan connection $C\Gamma$ associated with a Finsler manifold (M, L), $dim M \geqslant 3$, satisfies

$$R^r_{ijk} = k(g_{ij}\delta^r_k - g_{ik}\delta^r_j),$$

where k is a scalar function on $\mathcal{T} M$, positively homogeneous of degree zero ((0) p-homogeneous), then

(a) k is constant,
(b) if $k \neq 0$, then
 (1) the v-curvature of $C\Gamma$ vanishes: $S^r_{ijk} = 0$,
 (2) the hv-curvature of $C\Gamma$ is symmetric with respect to the last two indices: $P^r_{ijk} = P^r_{ikj}$.

In his paper [2], Hōjō showed, also by local calculations, that if the h-curvature R^r_{ijk} of the generalized Cartan connection $C\Gamma$, $dim M \geqslant 3$, satisfies[1]

$$R^r_{ijk} = k\mathfrak{A}_{j,k}\left\{q\, g_{ij}\delta^r_k + (q-2)(g_{ij}\ell_k\ell^r - \delta^r_j\ell_i\ell_k\right\},$$

where k is a (0) p-homogeneous scalar function and $1 \neq q \in \mathbb{R}$, then

(a) k is constant,
(b) if $k \neq 0$, then
 (1) the v-curvature of $C\Gamma$ satisfies $S^r_{ijk} = \frac{q-2}{2(1-q)}\mathfrak{A}_{j,k}\left\{\hbar_{ij}\hbar^r_k\right\}$,
 (2) the hv-curvature of $C\Gamma$ is symmetric with respect to the last two indices.

The aim of the present paper is to provide *intrinsic* proofs of Akbar-Zadeh's and Hōjō's theorems. As a by-product, some consequences concerning S_3-like and S_4-like spaces are deduced.

☆ ArXiv Number: 1201.2012 [math.DG].

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Peer review under responsibility of Egyptian Mathematical Society.

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[1] $\mathfrak{A}_{ij}$ indicates interchanges of indices j and k, and subtraction: $\mathfrak{A}_{ij}\{F_{ij}\} = F_{ij} - F_{ji}$.

http://dx.doi.org/10.1016/j.joems.2014.07.006
1110-256X

Thus the present work is formulated in a coordinate-free form, without being trapped into the complications of indices. Naturally, the coordinate expressions of the obtained results coincide with the starting local formulations.

1. Notation and preliminaries

In this section, we give a brief account of the basic concepts of the pullback approach to intrinsic Finsler geometry necessary for this work. For more details, we refer to [3–5]. We shall use the same notations as in [5].

In what follows, we denote by $\pi : \mathcal{T}M \to M$ the subbundle of nonzero vectors tangent to M and by $\mathfrak{X}(\pi(M))$ the $\mathfrak{F}(\mathcal{T}M)$-module of differentiable sections of the pullback bundle $\pi^{-1}(TM)$. The elements of $\mathfrak{X}(\pi(M))$ will be called π-vector fields and will be denoted by barred letters $\overline{X}$. The tensor fields on $\pi^{-1}(TM)$ will be called π-tensor fields. The fundamental π-vector field is the π-vector field $\bar{\eta}$ defined by $\bar{\eta}(u) = (u, u)$ for all $u \in \mathcal{T}M$.

We have the following short exact sequence of vector bundles

$$0 \to \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \pi^{-1}(TM) \to 0,$$

with the well known definitions of the bundle morphisms ρ and γ. The vector space $V_u(\mathcal{T}M) = \{X \in T_u(\mathcal{T}M) : d\pi(X) = 0\}$ is called the vertical space to M at u.

Let D be a linear connection (or simply a connection) on the pullback bundle $\pi^{-1}(TM)$. We associate with D the map $K : T\mathcal{T}M \to \pi^{-1}(TM) : X \mapsto D_X\bar{\eta}$, called the connection map of D. The vector space $H_u(\mathcal{T}M) = \{X \in T_u(\mathcal{T}M) : K(X) = 0\}$ is called the horizontal space to M at u. The connection D is said to be regular if $T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M) \forall u \in \mathcal{T}M$.

If M is endowed with a regular connection, then the vector bundle maps $\gamma, \rho|_{H(\mathcal{T}M)}$ and $K|_{V(\mathcal{T}M)}$ are vector bundle isomorphisms. The map $\beta := (\rho|_{H(\mathcal{T}M)})^{-1}$ will be called the horizontal map of the connection D.

The horizontal ((*h*)*h*-) and mixed ((*h*)*hv*-) torsion tensors of D, denoted by Q and T respectively, are defined by

$$Q(\overline{X}, \overline{Y}) = \mathbf{T}(\beta\overline{X}\beta\overline{Y}),\ T(\overline{X}, \overline{Y}) = \mathbf{T}(\gamma\overline{X}, \beta\overline{Y}) \quad \forall \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)),$$

where $\mathbf{T}$ is the torsion tensor field of D defined by

$$\mathbf{T}(X, Y) = D_X\rho Y - D_Y\rho X - \rho[X, Y] \quad \forall X, Y \in \mathfrak{X}(\mathcal{T}M).$$

The horizontal (*h*-), mixed (*hv*-) and vertical (*v*-) curvature tensors of D, denoted by R, P and S respectively, are defined by

$$R(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X}\beta\overline{Y})\overline{Z}, \quad P(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X}, \gamma\overline{Y})\overline{Z},$$
$$S(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\gamma\overline{X}, \gamma\overline{Y})\overline{Z},$$

where $\mathbf{K}$ is the (classical) curvature tensor field associated with D.

The contracted curvature tensors of D, denoted by $\widehat{R}$, $\widehat{P}$ and $\widehat{S}$ respectively, known also as the (*v*)*h*-, (*v*)*hv*- and (*v*)*v*-torsion tensors, are defined by

$$\widehat{R}(\overline{X}, \overline{Y}) = R(\overline{X}, \overline{Y})\bar{\eta}, \quad \widehat{P}(\overline{X}, \overline{Y}) = P(\overline{X}, \overline{Y})\bar{\eta},$$
$$\widehat{S}(\overline{X}, \overline{Y}) = S(\overline{X}, \overline{Y})\bar{\eta}.$$

If M is endowed with a metric g on $\pi^{-1}(TM)$, we write

$$R(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) := g(R(\overline{X}, \overline{Y})\overline{Z}, \overline{W}), \ldots, S(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) := g(S(\overline{X}, \overline{Y})\overline{Z}, \overline{W}). \tag{1.1}$$

The following result is of extreme importance.

Theorem 1.1 [6]. *Let (M, L) be a Finsler manifold and g the Finsler metric defined by L. There exists a unique regular connection ∇ on $\pi^{-1}(TM)$ such that*

(a) *∇ is metric : $\nabla g = 0$,*
(b) *The (h)h-torsion of ∇ vanishes: $Q = 0$,*
(c) *The (h)hv-torsion T of ∇ satisfies: $g(T(\overline{X}, \overline{Y}), \overline{Z}) = g(T(\overline{X}, \overline{Z}), \overline{Y})$.*

This connection is called the Cartan connection of the Finsler manifold (M, L).

2. First generalization of Akbar-Zadeh theorem

In this section, we investigate an intrinsic generalization of Akbar-Zadeh theorem. We begin first with the following two lemmas which will be useful for subsequent use.

Lemma 2.1. *Let ∇ be the Cartan connection of a Finsler manifold (M, L). For a π-tensor field ω of type $(1, 1)$, we have the following commutation formulae:*

(a) $\left(\overset{2}{\nabla}\overset{2}{\nabla}\omega\right)(\overline{X}, \overline{Y}, \overline{Z}) - \left(\overset{2}{\nabla}\overset{2}{\nabla}\omega\right)(\overline{Y}, \overline{X}, \overline{Z}) = \omega(S(\overline{X}, \overline{Y})\overline{Z}) - S(\overline{X}, \overline{Y})\omega(\overline{Z})$,

(b) $\left(\overset{2}{\nabla}\overset{1}{\nabla}\omega\right)(\overline{X}, \overline{Y}, \overline{Z}) - \left(\overset{1}{\nabla}\overset{2}{\nabla}\omega\right)(\overline{Y}, \overline{X}, \overline{Z}) = \omega(P(\overline{X}, \overline{Y})\overline{Z}) - P(\overline{X}, \overline{Y})\omega(\overline{Z}) + \left(\overset{2}{\nabla}\omega\right)(\widehat{P}(\overline{X}, \overline{Y}), \overline{Z}) + \left(\overset{1}{\nabla}\omega\right)(T(\overline{Y}, \overline{X}), \overline{Z})$,

(c) $\left(\overset{1}{\nabla}\overset{1}{\nabla}\omega\right)(\overline{X}, \overline{Y}, \overline{Z}) - \left(\overset{1}{\nabla}\overset{1}{\nabla}\omega\right)(\overline{Y}, \overline{X}, \overline{Z}) = \omega(R(\overline{X}, \overline{Y})\overline{Z}) - R(\overline{X}, \overline{Y})\omega(\overline{Z}) + \left(\overset{2}{\nabla}\omega\right)(\widehat{R}(\overline{X}, \overline{Y}), \overline{Z})$,

where $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ are the h- and v-covariant derivatives associated with ∇.

Lemma 2.2. *Let (M, L) be a Finsler manifold, g the Finsler metric defined by L, $\ell := L^{-1}i_{\bar{\eta}}g$ and $\hbar := g - \ell \circ \ell$ the angular metric tensor. Then we have:*

(a) $\overset{1}{\nabla} L = 0, \quad \overset{2}{\nabla} L = \ell$.
(b) $\overset{1}{\nabla} \ell = 0, \quad \overset{2}{\nabla} \ell = L^{-1}\hbar$.
(c) $i_{\bar{\eta}}\ell = L, \quad i_{\bar{\eta}}\hbar = 0$.

Proof. The assertions follow the facts that $\nabla g = 0$ and $g(\bar{\eta}, \bar{\eta}) = L^2$. □

Now, we have

Theorem 2.3. *Let (M, L) be a Finsler manifold of dimension n and g the Finsler metric defined by L. If the (v)h-torsion tensor $\widehat{R}$ of the Cartan connection is of the form*

$$\widehat{R}(\overline{X}, \overline{Y}) = kL(\ell(\overline{X})\overline{Y} - \ell(\overline{Y})\overline{X}), \tag{2.1}$$