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ORIGINAL ARTICLE

The quasi-uniform character of a topological semigroup

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Abstract The topological embedding of a topological semigroup S, commutative with the property of cancelation, into the group $G = S \times S/R$, (R the equivalence $(a,b)R(a',b') \iff ab' = a'b$) to which S is algebraically embedded, was the subject of the search for the mathematicians of a long period. It was based on the fact that S must naturally be a uniform topological space, as every topological group was. The present paper is devoted to the fact that a quasi-uniformity is defined to any topological space, thus to any topological semigroup.

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1. Introduction

1.1. In a series of papers for a long period the mathematicians engaged in *the embedding of a topological commutative semigroup with cancelation to a topological group*. The basic idea was very simple: since a topological group is a *uniform space*, that is a very nice space, it seems a natural demand for a topological semigroup, which embeds to a topological group, to be a uniform space as well. (Cf. the paper of E. Scheiferdecker [12, 1956] and the papers of [11,14,15,4,5,1,2,6] and others). In [3, 2001] the authors refer to a *quasi-uniformity* on a semigroup, that is: a topological semigroup S has a neutral element e and a neighborhood filter $\eta(e)$ of e which gives to S a *quasi-uniform structure*. On the other hand, the operations on the topological semigroups and groups must be continuous.

In the present paper we start with the *quasi-uniformity* which every topological T_0 structure has, hence every topological commutative with cancelation semigroup has. We suppose that the topology of the given *topological semigroup* is *weaker* or *equal* than the one which this structure may has. It is evident that if S is a semigroup and R is an *equivalence relation* on it, the quotient S/R is not a group, not even a semigroup. Meantime, it is defined the *specialization ordering* which has every T_0 but not T_1 topological space. The compatibility of the structures (of topology and of being the space semigroup) and the extension which Szpilran in [13] induces to an ordered space, seem to be obligatory for us.

1.2. In the remaining part of this paragraph we give necessary elements from the relative theory.

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A *semigroup* S is called *topological semigroup*, if there is a topology τ such that the function

$$\Phi : S \times S \to S, \Phi(x,y) = x \cdot y \quad \text{(or simply } = xy)$$

is continuous. A *group* G is called *topological group* if the functions Φ and K

$$\Phi : S \times S \to S, \Phi(x,y) = x \cdot y \quad \text{and} \quad K : G \to G, K(g) = g^{-1}$$

are continuous.

A *uniform space* on a set X is a filter $\mathbf{U}$ on $X \times X$ such that: (a) Each member of $\mathbf{U}$ contains the diagonal of $X \times X$. (b) If $U \in \mathbf{U}$, then $V \circ V \subseteq U$ for some $V \in \mathbf{U}$. (c) There is a base of $\mathbf{U}$ from symmetrical elements. The elements of $\mathbf{U}$ are called *entourages*.

If the structure lacks the condition (c), then the space is a *quasi-uniform*. In a semi-group S (resp. a *group* G) by $\tau(\mathbf{U})$ we denote the topology that originated by a *quasi-uniformity* (resp. a *uniformity*) $\mathbf{U}$. Also by $(S, \cdot, \tau(\mathbf{U}))$ we denote the whole structure.

Besides, W.J. Pervin (in [9]) in 1962, firstly published the statement: *"For every topological space there is a quasi-uniformity which induces the given topology"*. Pervin, in the above paper, says that for a topological space (X, τ), the sets

$$U_O = \{(O \times O) \cup [(X \setminus O) \times X] \mid O \in \tau\}$$

define a base for a *quasi-uniformity*, where $O \in \tau$. For every fixed O, the set U_O is an entourage of the quasi-uniformity.

1.3. The *quotient structure* (or *quotient semigroup*) $Q = Q(S, \Sigma)$, (Σ is a commutative sub-semigroup of S), is a set whose elements are of the form $a\alpha^{-1}, a \in S, \alpha \in \Sigma$. So $Q(S,\Sigma) = S \times \Sigma / R$, where R is an equivalence relation defined by: $(a,b)R(c,d) \iff ad = bc$, the operation in $S \times \Sigma$ is component-wise. If the semigroup S is commutative we can write $Q = Q(S,S)$ for the *quotient structure* and the structure $G = S \times S/R$, (R the known relation), is a group to which S is algebraically embedded. This topological embedding of S into the above G is exactly the object of the "embedding" which mathematicians made during the period we have referred to.

1.4. The authors of [3] define a *quasi-uniformity* for a topological commutative semigroup $(S, \cdot, \tau)$. The sets of the form

$$\overline{U} = \{(x,y) \mid y \in xU, U \in \eta(e)\}.$$

are the entourages of the space. The proof of this proposition is based on the fact that for every element U of the $\eta(e)$, there is another element V, such that $V \cdot V \subseteq U(e)$. On the other hand, this construction of a quasi-uniform space is compatible with the one introducing by Pervin.

1.5. In his classical paper [12], Scheiferdecker gave the notion of the *invariance for a uniformity* $\mathbf{U}$. Let $U \in \mathbf{U}$ and $a, b, k \in S$. Then

$$(a,b) \in U \iff (ka, kb) \in U.$$

The main theorem in [12] which we are interesting to, is the following:

1.6. Theorem (Scheiferdecker, [12, p. 375]). *Necessary and sufficient conditions for a topological semigroup $(S, \cdot, \tau)$ (τ the topology of S) to embed into its quotient group $G = S \times S/R$, where R is the known equivalence relation, are the following:*

(a) *The topology τ is the one induced by a uniformity $\mathbf{U}$.*
(b) *The uniform structure may be defined via entourages which fulfill the "invariance" property.* □

Scheiferdecker considered the above G and the structure $(S, \cdot, \tau = \tau(\mathbf{U}))$, where the topology $\tau(\mathbf{U})$ is the one that is induced from the uniformity of $\mathbf{U}$. He proved that the subsets

$$U_1 = \{(A,B) \in Q \times Q | (A = \alpha^{-1}a, B = \beta^{-1}b) \text{ and } (x\alpha = y\beta \in \Sigma \Rightarrow (xa, yb) \in U, U \in \mathbf{U})\}$$

$a, b \in S, \alpha, \beta \in \Sigma$, constitute the entourages of a new *uniformity*, whose the trace on S is the same topology τ. We denote this new uniformity by $\mathbf{U}_1$.

1.7. This paper is divided into 3 paragraphs. More precisely, in 1 the paper's preliminaries are given. In paragraph 2 we present the main part of this research. Especially we examine and investigate many properties of a topological semigroup, without considering the notion of the quasi-uniformity (see for example 2.2, 2.3, 2.5, 2.6, 2.8, etc.). Finally, paragraph 3 refers to the specialization inequality define on a T_0 and not a T_1 space.

2. Quasi-uniform structure in a semigroup

In the sequel, S is always a *commutative semigroup with cancelation*. The condition $aS \cap bS \neq \emptyset, a, b \in S$ ([8]), means that the equivalence relation R such that

$$(a,b)R(c,d) \iff ad = bc, a, b, c, d \in S,$$

is not void. We suppose that this condition is in valid through all the paper. The function

$$\pi : S \times S \to G, \pi((a,b)) = \overline{(a,b)}$$

assigns to each $(a,b) \in S \times S$ the equivalence class in G containing the element (a,b) and which we symbolize by $\overline{(a,b)}$.

2.1. Examples

(1) In the real line we consider additively the set $\Re$, (the set of real numbers), and as topology the one which has as base the intervals $(a, +\infty), a \in \Re$. The set $\Re$ is the set of symbols which finally we construct. We embed this in the set $G = \Re \times \Re/R, R$ the known equivalence relation, which is the natural construction of real numbers with the natural topology. The first topology is weakest of the second.
(2) The same problem in the interval $[0,1]$ with operation the multiplication, the numbers their-selves are the symbols we note and the topology, the one which has as base the set of the form $\{(a,1), a \in [0,1)\}$. It embeds into $G = [0,1) \times [0,1)/R$ of the natural construction of the set of real number and with the natural topology. The former topology is again weaker than the topology of G.
(3) If in 1. we consider as the first and the second topologies the Sorgenfrey topology of $\Re$ (the set of natural numbers) the results are the expected ones. The Sorgenfrey topology of $\Re$ which has as relation the couples: $\{(x,y) \mid x \leqslant y < x + \epsilon\}$.

2.2. Proposition. *If a quasi-uniformity $\mathbf{U}$ is defined on a commutative with cancelation semigroup $(S, \cdot)$ and has the property*

$$(\forall U \in \mathbf{U})(\forall a \in S)[U \subseteq (a,a)U],$$

then S is a topological semigroup.