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ORIGINAL ARTICLE

Coefficient estimates for a subclass of analytic and bi-univalent functions

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Abstract In the present investigation, we consider a new subclass $\Sigma(\tau,\gamma,\varphi)$ of the class Σ consisting of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. For functions belonging to the class $\Sigma(\tau,\gamma,\varphi)$ introduced here, we obtain estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. Several related classes of analytic and bi-univalent functions in $\mathbb{U}$ are also considered and connections to some of the earlier known results are pointed out.

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1. Introduction, definitions and preliminaries

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$ (see, for details [1,2]). Let $\mathcal{P}$ denote the family of functions $p(z)$, which are analytic in $\mathbb{U}$ such that

$$p(0) = 1 \quad \text{and} \quad \Re\big(p(z)\big) > 0 \quad (z \in \mathbb{U}).$$

An analytic function f is said to be subordinate to another analytic function g, written as

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function w, which is analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(w(z)\big).$$

In particular, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

It is known that, if $f(z)$ is an analytic univalent function from a domain $\mathbb{D}_1$ onto a domain $\mathbb{D}_2$, then the inverse function $g(z)$ defined by

$$g(f(z)) = z \quad (z \in \mathbb{D}_1)$$

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is an analytic and univalent mapping from $\mathbb{D}_2$ onto $\mathbb{D}_1$. Furthermore, it is well known by the familiar *Koebe One-Quarter Theorem* (see [1]) that the image of $\mathbb{U}$ under every function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$. Thus, clearly, every univalent function f in $\mathbb{U}$ has an inverse f^{-1} satisfying the following conditions:

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right).$$

The inverse of the function $f(z)$ has a series expansion in some disk about the origin of the form:

$$f^{-1}(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \cdots. \tag{1.2}$$

It was shown earlier (see [3,4]) that the inverse of the Koebe function provides the best bound for all $|\gamma_k|$ in (1.2). New proofs of this result, together with unexpected and unusual behavior of the coefficients γ_k in (1.2) for various subclasses of the univalent function class $\mathcal{S}$, have generated further interest in this problem (see, for details, [5–8]).

A function $f(z)$, which is univalent in a neighborhood of the origin, and its inverse $f^{-1}(w)$ satisfy the following condition:

$$f(f^{-1}(w)) = w$$

or, equivalently,

$$w = f^{-1}(w) + a_2[f^{-1}(w)]^2 + a_3[f^{-1}(w)]^3 + \cdots. \tag{1.3}$$

Using (1.1) and (1.2) in (1.3), we have

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots. \tag{1.4}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. We denote by Σ the class of all functions $f(z)$ which are bi-univalent in $\mathbb{U}$ and are given by the Taylor–Maclaurin series expansion (1.1).

The familiar Koebe function is not a member of Σ because it maps the unit disk $\mathbb{U}$ univalently onto the entire complex plane minus a slit along the line $-\frac{1}{4}$ to $-\infty$. Hence the image domain does not contain the unit disk $\mathbb{U}$.

In 1985 Louis de Branges [9] proved the celebrated *Bieberbach Conjecture* which states that, for each $f(z) \in \mathcal{S}$ given by the Taylor–Maclaurin series expansion (1.1), the following coefficient inequality holds true:

$$|a_n| \leqq n \quad (n \in \mathbb{N} \setminus \{1\}),$$

$\mathbb{N}$ being the set of positive integers. Lewin [10] investigated the class Σ of bi-univalent functions and, by using Grunsky inequalities, he showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [11] conjectured that $|a_2| \leqq \sqrt{2}$. Netanyahu [12], on the other hand, showed that (see also [13])

$$\max_{f \in \Sigma} |a_2| = \frac{4}{3}.$$

Later in 1981, Styer and Wright [14] showed that there are functions $f(z) \in \Sigma$ for which $|a_2| > \frac{4}{3}$. By considering the function $h_\theta(z)$ given by

$$h_\theta(z) := \left(\frac{ze^{-i\theta}}{1 - (ze^{-i\theta})^2}\right)\cos\theta + \left[\frac{i}{2}\log\left(\frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}}\right)\right]\sin\theta \quad \left(0 \leqq \theta < \frac{\pi}{2}\right),$$

so that, obviously, $h_\theta \in \mathcal{S}$, Styer and Wright [14] showed that, for θ sufficiently near $\frac{\pi}{2}$, $h_\theta \in \Sigma$. In the same year 1985, Tan [15] showed that $|a_2| \leqq 1.485$, which is the best known estimate for functions in the class Σ. The coefficient estimate problem involving the bound of $|a_n| (n \in \mathbb{N} \setminus \{1, 2\})$ for each $f \in \Sigma$ given by (1.1) is still an open problem.

For a further historical account of functions in the class Σ, see the work by Srivastava et al. [16] (see also [17,18]). In fact, judging by the remarkable flood of papers on non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ of various subclasses of the bi-univalent function class Σ (see, for example, [19–30,35,31–34,15,36–38]), the above-cited recent pioneering work of Srivastava et al. [16] has apparently revived the study of analytic and bi-univalent functions in recent years (see also [39,40]).

In the present investigation, we derive estimates on the initial coefficients $|a_2|$ and $|a_3|$ of a new subclass of the bi-univalent function class Σ. Several related classes are also considered and connections to earlier known results are made. The classes introduced in this paper are motivated by the corresponding classes investigated in [41–45].

Let φ be an analytic function with positive real part in $\mathbb{U}$ such that $\varphi(0) = 1, \varphi'(0) > 0$ and $\varphi(\mathbb{U})$ is symmetric with respect to the real axis. Such a function has a series expansion of the form:

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0). \tag{1.5}$$

We now introduce the following class of bi-univalent functions.

Definition 1. Let $0 \leqq \gamma \leqq 1$ and $\tau \in \mathbb{C} \setminus \{0\}$. A function $f \in \Sigma$ is said to be in the class $\Sigma(\tau, \gamma, \varphi)$ if each of the following subordination conditions holds true:

$$1 + \frac{1}{\tau}[f'(z) + \gamma z f''(z) - 1] \prec \varphi(z) \qquad (z \in \mathbb{U}) \tag{1.6}$$

and

$$1 + \frac{1}{\tau}[g'(w) + \gamma w g''(w) - 1] \prec \varphi(w) \qquad (w \in \mathbb{U}), \tag{1.7}$$

where $g(w) = f^{-1}(w)$.

In our investigation of the coefficient problem for functions in the class $\Sigma(\tau, \gamma, \varphi)$, we shall need the following lemma.

Lemma 1 (see [1]). *Let the function $p \in \mathcal{P}$ be given by the following series:*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \quad (z \in \mathbb{U}). \tag{1.8}$$

The sharp estimate given by

$$|c_n| \leqq 2 \qquad (n \in \mathbb{N}), \tag{1.9}$$

holds true.

2. A set of main results

For functions in the class $\Sigma(\tau, \gamma, \varphi)$, the following result is obtained.

Theorem 1. *Let $f(z) \in \Sigma(\tau, \gamma, \varphi)$ be of the form (1.1). Then*