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ORIGINAL ARTICLE

Solving systems of high-order linear differential–difference equations via Euler matrix method

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Abstract This paper contributes a new matrix method for solving systems of high-order linear differential–difference equations with variable coefficients under given initial conditions. On the basis of the presented approach, the matrix forms of the Euler polynomials and their derivatives are constructed, and then by substituting the collocation points into the matrix forms, the fundamental matrix equation is formed. This matrix equation corresponds to a system of linear algebraic equations. By solving this system, the unknown Euler coefficients are determined. Some illustrative examples with comparisons are given. The results demonstrate reliability and efficiency of the proposed method.

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1. Introduction

In the recent years, the systems of differential–difference equations [1], treated as models of some physical phenomena, have been received considerable attention. They are usually difficult to solve analytically; so a numerical method is needed. Recently, much attention has been given in the literature to the development, analysis, and implementation of methods for differential and differential–difference equations (see [2–14] for instance). In this research we try to introduce a solution of a system of high-order linear differential–difference equations in the form

$$\sum_{n=0}^{m}\sum_{j=1}^{k}F_{i,j}^{n}(x)y_j^{(n)}(\mu x+\lambda)=g_i(x),\quad a\leqslant x\leqslant b,\quad i=1,2,\ldots,k,\tag{1}$$

subject to the initial conditions

$$\sum_{j=0}^{m-1}(a_{i,j}^{n}y_n^{(j)}(a)+b_{i,j}^{n}y_n^{(j)}(b)+c_{i,j}^{n}y_n^{(j)}(c))=\mu_{n,i},$$
$$a\leqslant c\leqslant b,\quad i=0,1,\ldots,m-1,\quad n=1,\ldots,k,\tag{2}$$

where $a_{i,j}^n$, $b_{i,j}^n$, $c_{i,j}^n$, μ, λ and $\mu_{n,i}$ are real or complex constants, meanwhile $F_{i,j}^n(x)$, $g_i(x)$ are continuous functions defined on the interval $a\leqslant x\leqslant b$.

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2. Basic matrix relations for solution

The classical Euler polynomials $E_n(x)$ is usually defined as [15]

$$\sum_{k=0}^{n}\binom{n}{k}E_k(x)+E_n(x)=2x^n,\quad n\in N. \tag{3}$$

Let $X^T(x)$ be the $(N+1)\times 1$ matrix and P_{N+1} be the $(N+1)\times(N+1)$ lower triangular matrix defined by

$$X^T(x)=\left[1,x,\ldots,x^N\right]^T,\quad [P_{N+1}]_{ij}=\binom{i-1}{j-1},\quad i\geqslant j.$$

If n varies from 0 to N, the property (3) can be represented as matrix systems of equations

$$\frac{1}{2}(P_{N+1}+I_{N+1})E^T(x)=X^T(x).$$

Thus, the Euler vector can be given directly from

$$E^T(x)=D^{-1}X^T(x)\iff E(x)=X(x)(D^{-1})^T. \tag{4}$$

A relation between Euler polynomials and their derivatives is as follows $(E_n'(x)=nE_{n-1}(x),n=1,2,\ldots)$

$$\underbrace{[E_0(x),E_1(x),\ldots,E_N(x)]'}_{E'(x)}=\underbrace{[E_0(x),E_1(x),\ldots,E_N(x)]}_{E(x)}\underbrace{\begin{bmatrix}0&1&0&\ldots&0\\0&0&2&\ldots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\ldots&N\\0&0&0&\ldots&0\end{bmatrix}}_{M}. \tag{5}$$

We recall that, M is the Euler operational matrix of differentiation. Trivially $E^{(n)}(x)=E(x)M^n$ for all positive integers n, where our purpose from $E^{(n)}(x)$ is the nth derivative of $E(x)$. We can write $y_i(x)$ in the matrix form as follows;

$$y_i(x)=\sum_{n=0}^{N}a_{i,n}E_n(x)=E(x)A_i,\quad i=1,2,\ldots,k,$$
$$a\leqslant x\leqslant b, \tag{6}$$

where the Euler coefficient vector A_i and the Euler vector $E(x)$ are given by

$$A_i=[a_{i,0},a_{i,1},\ldots,a_{i,N}]^T,\quad E(x)=[E_0(x),E_1(x),\ldots,E_N(x)],$$

then the nth derivative of $y_i(x)$ can be expressed in the matrix form by

$$y_i^{(n)}(x)=E^{(n)}(x)A_i,\quad i=1,2,\ldots,k,\quad n=0,1,\ldots m. \tag{7}$$

Making use of (4), (5) and (7) yields

$$y_i^{(n)}(x)=E(x)M^nA_i=X(x)(D^{-1})^TM^nA_i,$$
$$i=1,2,\ldots,k,\quad n=0,1,\ldots m. \tag{8}$$

By putting $x\to\mu x+\lambda$ in the relation (8), we obtain the matrix form

$$y_i^{(n)}(\mu x+\lambda)=E(\mu x+\lambda)M^nA_i=X(\mu x+\lambda)(D^{-1})^TM^nA_i. \tag{9}$$

To obtain the matrix $X(\mu x+\lambda)$ in terms of the matrix $X(x)$, we can use the following relation:

$$X(\mu x+\lambda)=X(x)B(\mu,\lambda), \tag{10}$$

where

$$X(\mu x+\lambda)=\left[1,(\mu x+\lambda),(\mu x+\lambda)^2,\ldots,(\mu x+\lambda)^N\right],$$

for $\mu\neq 0$ and $\lambda\neq 0$,

$$B(\mu,\lambda)=\begin{bmatrix}\binom{0}{0}\mu^0\lambda^0&\binom{1}{0}\mu^0\lambda^1&\binom{2}{0}\mu^0\lambda^2&\ldots&\binom{N}{0}\mu^0\lambda^N\\0&\binom{1}{1}\mu^1\lambda^0&\binom{2}{1}\mu^1\lambda^1&\ldots&\binom{N}{1}\mu^1\lambda^{N-1}\\ \vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\ldots&\binom{N}{N}\mu^N\lambda^0\end{bmatrix}\text{ and for}$$

$$\mu\neq 0\text{ and }\lambda=0, B(\mu,\lambda)=\begin{bmatrix}\mu^0&0&0&\ldots&0\\0&\mu^1&0&\ldots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\ldots&\mu^N\end{bmatrix}.$$

By substituting the relation (10) into (9), we get

$$y_i^{(n)}(\mu x+\lambda)=X(x)B(\mu,\lambda)(D^{-1})^TM^nA_i,$$
$$i=1,2,\ldots,k,\quad n=0,1,\ldots m. \tag{11}$$

Hence, the matrices $y^{(n)}(\mu x+\lambda)$, $n=0,1,\ldots m$ can be expressed as follows

$$y^{(n)}(\mu x+\lambda)=\overline{X}(x)\overline{B}(\mu,\lambda)\overline{D}\overline{M}^nA, \tag{12}$$

where

$$y^{(n)}(\mu x+\lambda)=\begin{bmatrix}y_1^{(n)}(\mu x+\lambda)\\y_2^{(n)}(\mu x+\lambda)\\ \vdots\\y_k^{(n)}(\mu x+\lambda)\end{bmatrix},$$

$$\overline{X}(x)=\begin{bmatrix}X(x)&0&\cdots&0\\0&X(x)&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\0&0&\cdots&X(x)\end{bmatrix},$$

$$\overline{B}(\mu,\lambda)=\begin{bmatrix}B(\mu,\lambda)&0&\cdots&0\\0&B(\mu,\lambda)&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\0&0&\cdots&B(\mu,\lambda)\end{bmatrix},$$

$$\overline{D}=\begin{bmatrix}(D^{-1})^T&0&\cdots&0\\0&(D^{-1})^T&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\0&0&\cdots&(D^{-1})^T\end{bmatrix},$$

$$\overline{M}^n=\begin{bmatrix}M^n&0&\cdots&0\\0&M^n&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\0&0&\cdots&M^n\end{bmatrix},\quad A=\begin{bmatrix}A_1\\A_2\\ \vdots\\A_k\end{bmatrix}.$$

2.1. Matrix representation for the conditions

We can write Eq. (2) for $i=1,2,\ldots,k$ in the matrix form as $\sum_{j=0}^{m-1}[a_j^iy_i^{(j)}(a)+b_j^iy_i^{(j)}(b)+c_j^iy_i^{(j)}(c)]=[\mu_i]$, where