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ORIGINAL ARTICLE

On an explicit formula for inverse of triangular matrices

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Abstract In the present article, we define difference operators $B_L(a[m])$ and $B_U(a[m])$ which represent a lower triangular and upper triangular infinite matrices, respectively. In fact, the operators $B_L(a[m])$ and $B_U(a[m])$ are defined by $(B_L(a[m])x)_k = \sum_{i=0}^{m} a_{k-i}(i)x_{k-i}$ and $(B_U(a[m])x)_k = \sum_{i=0}^{m} a_{k+i}(i)x_{k+i}$ for all $k, m \in \mathbb{N}_0 = \{0,1,2,3,\ldots\}$, where $a[m] = \{a(0), a(1), \ldots a(m)\}$, the set of convergent sequences $a(i) = (a_k(i))_{k\in\mathbb{N}_0} (0 \leqslant i \leqslant m)$ of real numbers. Indeed, under different limiting conditions, both the operators unify most of the difference operators defined by various triangles such as $\Delta, \Delta^{(1)}, \Delta^m, \Delta^{(m)} (m \in \mathbb{N}_0), \Delta^\alpha, \Delta^{(\alpha)} (\alpha \in \mathbb{R}), B(r,s), B(r,s,t), B(\tilde{r},\tilde{s},\tilde{t},\tilde{u})$, and many others. Also, we derive an alternative method for finding the inverse of infinite matrices $B_L(a[m])$ and $B_U(a[m])$ and as an application of it we implement this idea to obtain the inverse of triangular matrices with finite support.

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1. Introduction, preliminaries and definitions

Difference operators are one of the important subclasses of Toeplitz operators where most of them are reduced to triangles under different limiting conditions. Triangular matrices have several applications in scientific computations and engineering, and the most useful contributions are solving the system of linear equations and finding spectral properties of bounded linear operators. Several methods have been employed to find the inverse of a triangle such as back ward substitution and elimination methods. The main idea of this note is to study certain triangles and derive an alternative method for their inverses.

Let w be the space all real valued sequences and for $m \in \mathbb{N}_0, a[m]$ be the set of convergent sequences $a(i) = (a_k(i))_{k\in\mathbb{N}_0} (0 \leqslant i \leqslant m)$ of real numbers. Let $x = (x_k)$ be any sequence in w, then we define the generalized difference operators $B_L(a[m])$ and $B_U(a[m])$ as :

$$(B_L(a[m])x)_k = a_k(0)x_k + a_{k-1}(1)x_{k-1} + a_{k-2}(2)x_{k-2} + \cdots + a_{k-m}(m)x_{k-m} \quad \text{and}$$
$$(B_U(a[m])x)_k = a_k(0)x_k + a_{k+1}(1)x_{k+1} + a_{k+2}(2)x_{k+2} + \cdots + a_{k+m}(m)x_{k+m} (k \in \mathbb{N}_0).$$

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It is being understood conventionally that any term with negative subscript is equal to zero. The operators $B_L(a[m])$ and $B_U(a[m])$ can be expressed as a lower triangular matrix (L_{nk}) and an upper triangular matrix (U_{nk}), respectively, where

$$(L_{nk}) = \begin{pmatrix} a_0(0) & 0 & 0 & \dots & 0 & 0 & \dots \\ a_0(1) & a_1(0) & 0 & \dots & 0 & 0 & \dots \\ a_0(2) & a_1(1) & a_2(0) & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_0(m) & a_1(m-1) & a_2(m-2) & \dots & a_m(0) & 0 & \dots \\ 0 & a_1(m) & a_2(m-1) & \dots & a_m(1) & a_{m+1}(0) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and

$$(U_{nk}) = \begin{pmatrix} a_0(0) & a_0(1) & a_0(2) & \dots & a_0(m) & 0 & \dots \\ 0 & a_1(0) & a_1(1) & \dots & a_1(m-1) & a_1(m) & \dots \\ 0 & 0 & a_2(0) & \dots & a_2(m-2) & a_2(m-1) & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_m(0) & a_m(1) & \dots \\ 0 & 0 & 0 & \dots & 0 & a_{m+1}(0) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Different kinds of triangles via difference operators have been studied by various authors. For instance, triangles such as double banded Δ, triple banded $B(r,s,t)$, fourth banded $B(\tilde{r},\tilde{s},\tilde{t},\tilde{u})$, and $(m+1)$ banded Δ^m matrices have been introduced by Kızmaz [1], Furkan et al. [2], Dutta and Baliarsingh [3] and Et and Çolak [4], respectively. Altay and Başar [5] and Dutta and Baliarsingh [6] have studied the spectral properties of difference operators $B(r,s)$ and Δ^2, respectively. In fact, the detailed study of these operators involving topological properties, duals, matrix transformations and spectral properties is only possible by determining their inverse operators. Recently, Baliarsingh [7] and Dutta and Baliarsingh [8] have introduced fractional order difference matrix Δ^α and $(m+1)$sequential band matrix $B(a[m])$ and derived their corresponding inverse operators. However, the explicit formula for inverse of the lower triangle $B(a[m])$ has been employed in [8]. In fact, in that article this result has been proved by using counter examples, but in this investigation, we demonstrate these results in a more general way and extend those to upper triangular matrices.

Now, we define certain triangles generated by various means of the sequence $x=(x_k)$. Let $\mathcal{U}$ be the set of all sequences $u=(u_k)$ of real numbers such that $u_k \neq 0$ for all $k \in \mathbb{N}_0$. Let $r=(r_k), s=(s_k)$, and $t=(t_k)$ be three sequences in $\mathcal{U}$ and

$$T_n := \sum_{k=0}^{n} t_k (n \in \mathbb{N}_0).$$

Then the Cesàro mean of order one and Riesz mean with respect to the sequence $t=(t_k)$ are defined by the matrices $\mathcal{C}_1=(c_{nk})$ and $R^t=(r^t_{nk})$, respectively (see [9,10]), where

$$c_{nk} := \begin{cases} \frac{1}{n+1}, & (0 \leqslant k \leqslant n) \\ 0, & (k>n) \end{cases}, \quad (n,k \in \mathbb{N}_0)$$

$$r^t_{nk} := \begin{cases} \frac{t_n}{T_n}, & (0 \leqslant k \leqslant n) \\ 0, & (k>n) \end{cases}, \quad (n,k \in \mathbb{N}_0).$$

The generalized mean of the sequence $x=(x_k)$ can be computed by using $A(r,s,t)-$transform of x (see [11]), where $A(r,s,t)$ represents an infinite matrix a_{nk} and

$$a_{nk} := \begin{cases} \frac{s_{n-k}t_k}{r_n}, & (0 \leqslant k \leqslant n) \\ 0, & (k>n) \end{cases}, \quad (n,k \in \mathbb{N}_0).$$

2. Main results

In this section, we study certain results concerning the linearity, boundedness, and inverse properties of the infinite difference matrices $B_L(a[m])$ and $B_U(a[m])$. However, the results are valid for a matrix of infinite order, but it is convenient to implement those for the matrices of finite order.

Theorem 1. *The operators $B_L(a[m])$ and $B_U(a[m])$ defined from w to w are bounded linear operators.*

Proof. Linearity of the operators $B_L(a[m])$ and $B_U(a[m])$ are obvious and for boundedness,

$$\|B_L(a[m])\| = \|B_U(a[m])\| = \sup_{(k \in \mathbb{N}_0, 0 \leqslant i \leqslant m)} (m+1)|a_k(i)|. \quad \square$$

Theorem 2. [8], Theorem 2*If $a_k(0) \neq 0$ for all $k \in \mathbb{N}_0$, then an explicit formula for inverse of the difference operator $B_L(a[m])$ is given by*

$$L^{-1}_{nk} = \begin{cases} \frac{1}{a_n(0)}, & (k=n) \\ \frac{(-1)^{n-k}}{\prod_{j=k}^{n} a_j(0)} D^{(k)}_{n-k}(a[m]), & (0 \leqslant k \leqslant n-1), \quad (n,k \in \mathbb{N}_0). \\ 0, & (k>n) \end{cases}$$

where

$$D^{(k)}_n(a[m]) = \begin{vmatrix} a_k(1) & a_{k+1}(0) & 0 & \dots & 0 & 0 & \dots & 0 \\ a_k(2) & a_{k+1}(1) & a_{k+2}(0) & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_k(m) & a_{k+1}(m-1) & a_{k+2}(m-2) & \dots & a_{m-1}(1) & a_m(0) & \dots & 0 \\ 0 & a_{k+1}(m) & a_{k+2}(m) & \dots & a_{m-1}(2) & a_m(1) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n-m}(m) & \dots & \dots & \dots & a_{n+k-1}(1) \end{vmatrix}, \quad (n \geqslant 1).$$