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REVIEW PAPER

Bounded linear operators in quasi-normed linear space☆

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Abstract In this paper, we define continuity and boundedness of linear operators in quasi-normed linear space. Quasi-norm linear space of bounded linear operators is deduced. Concept of dual space is developed.

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Contents

0. Introduction

It is well known that metric and norm structures play a pivotal role in functional analysis. So in order to develop functional analysis one has to take care of the suitable generalization of these structures. Historically, the problem of generalization of the metric structure came first. Different authors introduced ideas of quasi-metric space [1,2] generalized metric space [3,4], generalized quasi-metric space [5], dislocated metric space [6], fuzzy metric space [7–9], statistical metric space [10], two

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metric space [11], quasi-normed linear space[12], fuzzy normed linear space [13], fuzzy Banach space[14], etc. Many authors [15–21] study the stability of different types of functional equations in different directions.

In [2], Rano introduce the concepts of Cauchy sequence, Convergent sequence, Open set, Closed set, etc. in a quasi-metric space and established some basic theorems such as Cantor's intersection theorem and Baire's category theorem in complete quasi-metric spaces. We give the definition of Contraction mapping and established some fixed point theorem with uniqueness. In [13], some results on finite dimensional quasi-normed linear spaces are established, the idea of equivalent quasi-norm is introduced and Riesz's lemma is proved in this space.

In this paper, we define continuity and boundedness of linear operators in quasi-normed linear space. Quasi-norm linear space of bounded linear operators is deduced. Concept of dual space is developed.

The organization of the paper is as follows:

In Section 1, comprises some preliminary results.

In Section 2, we introduce the concept of continuity and boundedness of linear operators in quasi-normed linear space.

Space of bounded linear operators and dual space are developed in Section 3.

In Section 4, we give some interesting open problems.

Throughout this paper straightforward proofs are omitted.

1. Some preliminary results

Definition 1.1 [12]. Let X be a linear space over the field F and θ the origin of X. Let $|\cdot|_q : X \to [0,\infty)$ satisfying the following conditions:

(QN-1) $|x|_q = 0$ iff $x = \theta$;
(QN-2) $|ex|_q = |e||x|_q$ for $x \in X$ and $e \in F$;
(QN-3) there exists a $K \geqslant 1$ such that

$$|x+y|_q \leqslant K\{|x|_q + |y|_q\} \quad \text{for } x, y \in X.$$

Then $(X, |\cdot|_q)$ is called a quasi-normed linear space (**qnls**) and the least value of the constant $K \geqslant 1$ is called the index of the quasi-norm $|\cdot|_q$.

The quasi-normed linear space $(X, |\cdot|_q)$ is called a strong quasi-normed linear space (**sqnls**) if it satisfies the following additional condition:

(QN-4) There exists $K \geqslant 1$ such that

$$\left|\sum_{i=1}^{n} x_i\right|_q \leqslant K\left\{\sum_{i=1}^{n} |x_i|_q\right\} \quad \forall x_i \in X, \quad \forall n \in N.$$

Note 1.1 [12]. In a quasi-normed linear space $(X, |\cdot|_q)$ with quasi index K,

$$\left|\sum_{i=1}^{n} x_i\right|_q \leqslant K^{n-1}\left\{\sum_{i=1}^{n} |x_i|_q\right\} \quad \forall x_i \in X, \quad \forall n \in N.$$

Note 1.2 [12]. If $K = 1$ then the quasi-norm $|\cdot|_q$ is reduced to a norm on X and $(X, |\cdot|_q)$ a normed linear space.

Note 1.3 [12]. Every normed linear space is a quasi-normed linear space but not conversely, which is justified by the following examples.

Example 1.1 [12]. Let $X = R^2$ be a linear space. For $x = (x_1, x_2) \in X$ define

$$|x|_q = \left(\sqrt{|x_1|} + \sqrt{|x_2|}\right)^2.$$

Then $(X, |\cdot|_q)$ is a quasi-normed linear space but not a normed linear space.

Definition 1.2 [12]. Let $(X, |\cdot|_q)$ be a quasi-normed linear space.

(i) A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is said
  (a) to converge to $x \in X$ denoted by $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} |x_n - x|_q = 0$;
  (b) to be a Cauchy sequence if $\lim_{m,n\to\infty} |x_n - x_m|_q = 0$.

(ii) A subset $B \subset X$ is said to be complete if every Cauchy sequence in B converges in B.
(iii) A subset A of X is said to be bounded if there exists a real number $M > 0$ such that $|x|_q \leqslant M\ \forall x \in A$.
(iv) A subset A of X is said to be closed if for any sequence $\{x_n\}$ of points of A with $\lim_{n\to\infty} x_n = x$ implies $x \in A$.
(v) A subset A of X is said to be compact if for any sequence $\{x_n\}$ of points of A has a convergent subsequence which converges to a point in A.

Proposition 1.1 [12]. *Let $(X, |\cdot|_q)$ be a quasi-normed linear space. Then*

(a) *the limit of a sequence $\{x_n\}$ in X if exists is unique;*
(b) *every subsequence of a convergent sequence converges to the same limit;*
(c) *every convergent sequence in X is a Cauchy sequence.*

Lemma 1.1 [12]. *Let $\{x_1, x_2, x_3, \ldots, x_n\}$ be a linearly independent set of vectors in a quasi-normed linear space $(X, |\cdot|_q)$. Then $\exists C > 0$ such that for any choice of scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have*

$$|\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n|_q \geqslant C(|\lambda_1| + |\lambda_2| + \cdots + |\lambda_n|).$$

Definition 1.3 [12]. Let $(X, |\cdot|_q)$ be a quasi-normed linear space. If X is a finite dimensional linear space then $(X, |\cdot|_q)$ is called a finite dimensional quasi-normed linear space.

2. Bounded linear operators in quasi-normed linear space

In this section we define continuous and bounded linear operators in quasi-normed linear spaces and study some properties in this space.

Definition 2.1. Let $(X_1, |\cdot|_{q_1})$ and $(X_2, |\cdot|_{q_2})$ be two quasi-normed linear spaces and $T : X_1 \to X_2$ be an operator. Then T is said to be continuous at $x \in X_1$ if for any sequence $\{x_n\}$ of