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ORIGINAL ARTICLE

The approximation of common element for maximal monotone operator, generalized mixed equilibrium problem and fixed point problem

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Abstract The purpose of this paper is to get strong convergence theorems for a countable family of relatively quasi-nonexpansive mappings $\{S_n\}_{n=0}^{\infty}$, a maximal monotone operator T, and a generalized mixed equilibrium problem in a uniformly smooth and uniformly convex Banach space lacking condition UARC. Two examples are given to support our results. One is a countable family of uniformly closed relatively quasi-nonexpansive mappings but not a countable family of relatively nonexpansive mappings. Another is uniformly closed but not satisfies condition UARC. Many recent results in this field have been unified and improved.

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1. Introduction

In an infinite-dimensional Hilbert space, Mann iterative algorithm has only weak covergence, in general, even for nonexpansive mappings. Hence in order to have strong convergence, in recent years, the hybrid iteration methods for approximating fixed points of nonlinear mappings have been introduced and studied by various authors [1–6].

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Let E be a smooth Banach space. We denote by ϕ the functional on $E \times E$ defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, J(y)\rangle + \|y\|^2, \quad \forall\ x,y \in E.$$

A point $p \in C$ is said to be an (strong) asymptotic fixed point of T if there exists a sequence $\{x_n\}_{n=0}^{\infty} \subset C$ such that $(x_n \rightarrow p)$ $x_n \rightharpoonup p$ and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of (strong) asymptotic fixed point is denoted by $(\widetilde{F}(T))$ $\widehat{F}(T)$. Let E be a smooth Banach space, we say that a mapping T is (weak) relatively nonexpansive (see [7–11]) if the following conditions are satisfied:

(i) $F(T) \neq \emptyset$;
(ii) $\phi(p, Tx) \leqslant \phi(p, x),\ \forall x \in C,\ p \in F(T)$;
(iii) $(F(T) = \widetilde{F}(T))$ $F(T) = \widehat{F}(T)$.

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A multivalued operator $T : E \to 2^{E^*}$ with domain $D(T) = \{z \in E : Tz \neq \emptyset\}$ is called monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \geqslant 0$ for each $x_i \in D(T)$ and $y_i \in Tx_i, i = 1, 2$. A monotone operator T is called maximal if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator. A method for solving the inclusion $0 \in Tx$ is the proximal point algorithm. This algorithm was first presented by Martinet [12] and generally studied by Rockafellar [13] in a Hilbert space. A mapping $A : C \to E^*$ is called α-inverse-strongly monotone, if there exists an $\alpha > 0$ such that $\langle Ax - Ay, x - y \rangle \geqslant \alpha \|Ax - Ay\|^2, \ \forall x, y \in C$.

It is easy to see that if $A : C \to E^*$ is an α-inverse-strongly monotone mapping, then it is $1/\alpha$-Lipschitzian. Let $T : E \to 2^{E^*}$ be a maximal monotone operator in a smooth Banach space E. We denote the resolvent of T by $J_r := (J + rT)^{-1}J$ for each $r > 0$. Then $J_r : E \to D(T)$ is a single-valued mapping. Also, $T^{-1}0 = F(J_r)$ for each $r > 0$, where $F(J_r)$ is the set of fixed points of J_r. For each $r > 0$, the Yosida approximation of T is defined by $A_r = (J - JJ_r)/r$. It is known that

$$A_r x \in T(J_r x), \quad \forall r > 0 \quad \text{and} \quad x \in E.$$

Let $\varphi : C \to R$ be a real-valued function and $A : C \to E^*$ be a nonlinear mapping and $f : C \times C \to R$ be a bifunction. For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y \in C, \lim_{t \to 0} f(tz + (1 - t)x, y) \leqslant f(x, y)$;
(A4) for each $x \in C, y \mapsto f(x, y)$ is convex and lower semi-continuous.

The generalized mixed equilibrium problem is to find $u \in C$ [14–16] such that:

$$f(u, y) + \varphi(y) - \varphi(u) + \langle Au, y - u \rangle \geqslant 0, \quad \forall \ y \in C. \tag{1.7}$$

Throughout this paper, we denote $f(u, y) + \varphi(y) - \varphi(u) + \langle Au, y - u \rangle$ by $F(x, y)$. The set of solutions of (1.7) is denoted by $GMEP(F, \varphi)$, i.e.,

$$GMEP(F, \varphi) = \{u \in C : f(u, y) + \varphi(y) - \varphi(u) + \langle Au, y - u \rangle \geqslant 0, \quad \forall \ y \in C\}.$$

If $A = 0$, then problem (1.7) is equivalent to mixed equilibrium problem studied by many authors, which is to find $u \in C$ such that

$$f(u, y) + \varphi(y) - \varphi(u) \geqslant 0, \quad \forall \ y \in C.$$

If $\varphi = 0$, then problem (1.7) is equivalent to generalized equilibrium problem considered by many authors, which is to find $u \in C$ such that

$$f(u, y) + \langle Au, y - u \rangle \geqslant 0, \quad \forall \ y \in C.$$

If $\varphi = 0, A = 0$, then problem (1.7) is reduces to equilibrium problem considered by many authors, which is to find $u \in C$ such that $f(u, y) \geqslant 0, \ \forall y \in C$.

The generalized mixed equilibrium problem includes fixed point problem, optimization problem, variational inequality problem, minimax problem, Nash equilibrium problem as spacial cases [17]. Some methods have been proposed to find its solutions. And, numerous problems in physics, optimation and economics can be reduced to find a solution of generalized equilibrium problem [18].

Algorithms for obtaining fixed point of relatively nonexpansive mappings have been studied widely. For instance, Mann iterative method, Ishikawa-type iterative method, Halpern-type iterative method, hybrid methods, and many other modified methods. Recently, utilizing Nakajo and Takahashi's idea [19], Qin and Su [20] introduced one iterative algorithm for a relatively nonexpansive mapping. By combining Kamimura and Takahashi's idea [21] with Qin and Su [20], Ceng et al. [22] introduced a hybrid proximal-type algorithm for finding an element of fixed point set and zero point set in a uniformly smooth and uniformly convex Banach space. In 2011, Ceng et al. [23] introduced and investigated one hybrid shrinking projection method for a generalized equilibrium problem, a maximal monotone operator and a countable family of relatively nonexpansive mappings. The authors obtained strong convergence theorems.

2. Preliminaries and lemmas

Let E be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty closed convex subset of E. It is well known that the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) = \arg\min_{y \in C} \phi(y, x), \quad \forall \ x \in E.$$

The existence and uniqueness of Π_C follows from the property of the functional $\phi(x, y)$ and strict monotonicity of the mapping J. And it is obvious that

$$(\|x\| - \|y\|)^2 \leqslant \phi(x, y) \leqslant (\|x\| + \|y\|)^2, \quad \forall \ x, y \in E.$$

Next, we recall the notion of generalized f-projection operator and its properties. Let $G : C \times E^* \to R \cup \{+\infty\}$ be a functional defined as following:

$$G(\xi, \varphi) = \|\xi\|^2 - 2\langle \xi, \varphi \rangle + \|\varphi\|^2 + 2\rho f(\xi), \tag{2.1}$$

where $\xi \in C, \varphi \in E^*, \rho$ is a positive number and $f : C \to R \cup \{+\infty\}$ is proper, convex and lower semi-continuous. From the definitions of G and f, it is easy to see the following properties:

(i) $G(\xi, \varphi)$ is convex and continuous with respect to φ when ξ is fixed.
(ii) $G(\xi, \varphi)$ is convex and lower semi-continuous relate to ξ when φ is fixed.

We can see that the functional G is a generalization of functional ϕ. That is, functional ϕ is a special case of functional G when $f \equiv 0$.

Definition 2.1 [24]. Let E be a real Banach space with its dual E^*. Let C be a nonempty, closed and convex subset of E. We say that $\Pi_C^f : E^* \to 2^C$ is *a generalized f-projection operator* if for any $\varphi \in E^*$,

$$\Pi_C^f \varphi = \{u \in C : G(u, \varphi) = \inf_{\xi \in C} G(\xi, \varphi)\}.$$

For the generalized f-projection operator, Wu and Huang [20] proved the following basic properties: