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ORIGINAL ARTICLE

Unified fixed point theorems for mappings in fuzzy metric spaces via implicit relations

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Abstract In this paper, we prove some fixed point theorems for weakly compatible mappings in fuzzy metric spaces employing common limit range property with implicit relations. We also furnish some illustrative examples to support our main results. As an application to our main result, we derive a fixed point theorem for four finite families of self-mappings which can be utilized to derive common fixed point theorems involving any finite number of mappings. Our results improve and extend a host of previously known results including the ones contained in the paper of Gopal et al. (2011).

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1. Introduction

In 1965, Zadeh [1] introduced the well-known concept of a fuzzy set in his seminal paper. In the last two decades there has been a tremendous development and growth in fuzzy mathematics. In 1975, Kramosil and Michálek [2] introduced the concept of fuzzy metric space, which opened an avenue for further development of analysis in such spaces. Further, George and Veeramani [3] modified the concept of fuzzy metric space introduced by Kramosil and Michálek [2] with a view to obtain a Hausdoroff topology on it. Fuzzy set theory has applications in applied sciences such as neural network theory, stability theory, mathematical programming, modeling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, and communication.

Mishra et al. [4] extended the notion of compatible mappings to fuzzy metric spaces and proved common fixed point theorems in presence the of continuity of at least one of the mappings, completeness of the underlying space and

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containment of the ranges among involved mappings. Further, Singh and Jain [5] weakened the notion of compatibility by using the notion of weakly compatible mappings in fuzzy metric spaces and showed that every pair of compatible mappings is weakly compatible but reverse is not true. Many mathematicians used different conditions on self-mappings and proved several fixed point theorems for contractions in fuzzy metric spaces (see [6–16]). However, the study of common fixed points of non-compatible mappings is also of great interest due to Pant [17]. In 2002, Aamri and Moutawakil [18] defined a property (E.A) for self-mappings which contained the class of non-compatible mappings in metric spaces. In a paper of Ali and Imdad [19], it was pointed out that property (E.A) allows replacing the completeness requirement of the space with a more natural condition of closedness of the range. Afterward, Liu et al. [20] defined a new property which contains the property (E.A) and proved some common fixed point theorems under hybrid contractive conditions. It was observed that the notion of common property (E.A) relatively relaxes the required containment of the range of one mapping into the range of other which is utilized to construct the sequence of joint iterates. Subsequently, there are a number of results proved for contraction mappings satisfying property (E.A) and common property (E.A) in fuzzy metric spaces (see [21–28]). In 2011, Sintunavarat and Kumam [29] coined the idea of "common limit range property" (also see [30–35]) which relaxes the condition of closedness of the underlying subspace. Recently, Imdad et al. [36] extended the notion of common limit range property to two pairs of self-mappings which relaxes the requirement on closedness of the subspaces. Several common fixed point theorems have been proved by many researcher in framework of fuzzy metric spaces via implicit relations (see [5,21,37]).

In fixed point theory, implicit relations are utilized to cover several contraction conditions in one go rather than proving a separate theorem for each contraction condition. In 2005, Singh and Jain [5] proved common fixed point theorems for semi-compatible mappings in fuzzy metric spaces satisfying an implicit function. Recently, Gopal et al. [24] defined two independent classes of implicit functions and obtained some fixed point results for two pairs of weakly compatible mappings satisfying common property (E.A).

In this paper, utilizing the implicit functions of Gopal et al. [24], we prove fixed point theorems for two pairs of weakly compatible mappings employing common limit range property. In process, many known results (especially the ones contained in Gopal et al. [24]) are enriched and improved. Some related results are also derived besides furnishing illustrative examples.

2. Preliminaries

Definition 2.1 [38]. A binary operation $*: [0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a*1=a$ for every $a\in[0,1]$,
(4) $a*b\leqslant c*d$ if $a\leqslant c$ and $b\leqslant d$ for all $a,b,c,d\in[0,1]$.

Three typical examples of continuous t-norms are minimum t-norm, that is, $a*b=\min\{a,b\}$, product t-norm, that is, $a*b=ab$ and Lukasievicz t-norm, that is, $a*b=\max\{a+b-1,0\}$.

Definition 2.2 [39]. Let X be any set. A fuzzy set in X is a function with domain X and values in $[0,1]$.

Definition 2.3 [3]. A triplet $(X,M,*)$ is a fuzzy metric space whenever X is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set on $X\times X\times(0,+\infty)$ satisfying the following conditions: for every $x,y,z\in X$ and $s,t>0$

(1) $M(x,y,t)>0$,
(2) $M(x,y,t)=1$ if and only if $x=y$,
(3) $M(x,y,t)=M(y,x,t)$,
(4) $M(x,z,t+s)\geqslant M(x,y,t)*M(y,z,s)$,
(5) $M(x,y,\cdot):(0,+\infty)\to(0,1]$ is continuous.

Note that $M(x,y,t)$ can be realized as the measure of nearness between x and y with respect to t. It is known that $M(x,y,\cdot)$ is non-decreasing for all $x,y\in X$. Let $(X,M,*)$ be a fuzzy metric space. For $t>0$, the open ball $\mathcal{B}(x,r,t)$ with center $x\in X$ and radius $0<r<1$ is defined by $\mathcal{B}(x,r,t)=\{y\in X: M(x,y,t)>1-r\}$. Now, the collection $\{\mathcal{B}(x,r,t): x\in X, 0<r<1, t>0\}$ is a neighborhood system for a topology τ on X induced by the fuzzy metric M. This topology is Hausdorff and first countable.

Definition 2.4 [3]. A sequence $\{x_n\}$ in X converges to x if and only if for each $\epsilon>0$ and each $t>0$, there exists $n_0\in\mathbb{N}$ such that $M(x_n,x,t)>1-\epsilon$ for all $n\geqslant n_0$.

In the following example, we know that every metric induces a fuzzy metric:

Example 2.1 [3]. Let (X,d) be a metric space. We define $a*b=ab$ for all $a,b\in[0,1]$ and let M_d be a fuzzy set on $X^2\times(0,+\infty)$ defined as follows:

$$M_d(x,y,t)=\frac{t}{t+d(x,y)}.$$

Then $(X,M_d,*)$ is a fuzzy metric space and the fuzzy metric M induced by the metric d is often referred to as the standard fuzzy metric. The fuzzy metric space $(X,M_d,*)$ is complete if and only if the metric space (X,d) is complete.

Definition 2.5 [4]. A pair (A,S) of self-mappings of a fuzzy metric space $(X,M,*)$ is said to be compatible if and only if $M(ASx_n,SAx_n,t)\to 1$ for all $t>0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n,Sx_n\to z$ for some $z\in X$ as $n\to\infty$.

Definition 2.6 [40]. A pair (A,S) of self-mappings of a non-empty set X is said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if $Az=Sz$ some $z\in X$, then $ASz=SAz$.

Remark 2.1 [40]. Two compatible self-mappings are weakly compatible, but the converse is not true. Therefore the concept of weak compatibility is more general than that of compatibility.