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ORIGINAL ARTICLE

A new generalized Weibull distribution generated by gamma random variables

Fredy Castellares [a], Artur J. Lemonte [b,*]

[a] *Departamento de Estatística, Universidade Federal de Minas Gerais, Belo Horizonte, MG, Brazil*
[b] *Departamento de Estatística, Universidade Federal de Pernambuco, Recife, PE, Brazil*

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Abstract We propose a new and simple representation for the probability density function of the gamma-G family of distributions as an absolutely convergent power series of the cumulative function of the baseline G distribution. Additionally, the special case the so-called gamma exponentiated Weibull model is introduced and studied in details.

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1. Introduction

The two-parameter Weibull distribution is a very popular distribution that has been extensively used over the past decades for modeling data in reliability, engineering and biological studies. It is well-known that the major weakness of the Weibull distribution is its inability to accommodate nonmonotone failure rates. The first generalization of the two-parameter Weibull distribution to accommodate nonmonotone failure rates was introduced by [1] and it is known as the exponentiated Weibull (EW) distribution. The three-parameter EW distribution has cumulative function in the form $G_{\rm EW}(x) = G_{\rm EW}(x; b, \alpha, \beta) = (1 - e^{-\alpha x^{\beta}})^{b}$, $x > 0$, where $\beta > 0$ and $b > 0$ are shape parameters, and $\alpha > 0$ is the scale parameter. The EW density function is $g_{\rm EW}(x) = g_{\rm EW}(x; b, \alpha, \beta) = b\,\alpha\,\beta\,x^{\beta-1} e^{-\alpha x^{\beta}}(1 - e^{-\alpha x^{\beta}})^{b-1}$, $x > 0$. The reader is refereed to [2] for an overview of the EW distribution.

The recent literature has suggested several other ways of extending well-known distributions. The earliest is the class of distributions generated by a standard beta random variable introduced by [3]. The more recent ones are as follows: the class of distributions generated by [4]'s random variable introduced by [5]; the class of distributions generated by gamma random variables introduced by [6,7]; the class of distributions generated by [8]'s generalized beta random variable introduced by [9]; and the *T-X* family of distributions introduced in [10]. Some of the above methods were recently discussed in [11]. By using the generator approach suggested by [3], several generalized distributions have been proposed in the last few years. In particular, [3,12,13] defined the beta normal, beta Fréchet, beta Gumbel, beta exponential, beta Weibull and beta Pareto distributions by taking $G(x)$ to be the cumulative function of the normal, Fréchet, Gumbel, exponential, Weibull and Pareto distributions, respectively. More recently, [14–20] defined the

* Corresponding author.
E-mail address: arturlemonte@gmail.com (A.J. Lemonte).

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beta generalized exponential, beta generalized half-normal, beta modified Weibull, beta Burr XII, beta Birnbaum–Saunders, beta Laplace and beta half-Cauchy distributions, respectively. Some generalized distributions generated by [4]'s random variable are proposed in [21–23]. Recently, a five-parameter continuous model generated by [8]'s generalized beta random variable was proposed by [24]. As can be observed from these references, several new generalized distributions were constructed from the logit of a beta random variable. On the other hand, the generator approaches in [5–7,9,10] have not been much explored for generating new classes of generalized distributions. We refer the reader to [25,26] for some generalized distributions constructed by using the generator approach of [10].

In this paper, we use the generator approach of [6] to introduce a new generalized Weibull family of distributions. The generator approach introduced by these authors is as follows. For any continuous baseline cumulative distribution function (cdf) $G(x) = G(x;\boldsymbol{\tau})$ and parameter vector $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_q)^\top$ of dimension q, the cumulative function of the new distribution is defined by $F(x) = F(x;a,\boldsymbol{\tau}) = \Gamma(a)^{-1}\gamma(a, -\log[1-G(x)])$, $x \in \mathbb{R}$, where $a > 0$ is an additional shape parameter to those in $\boldsymbol{\tau}$ that aims to introduce skewness and to provide greater flexibility of its tails. Also, $\Gamma(r) = \int_0^\infty t^{r-1}\mathrm{e}^{-t}\mathrm{d}t$ is the gamma function, and $\gamma(r,s) = \int_0^s t^{r-1}\mathrm{e}^{-t}\mathrm{d}t$ is the incomplete gamma function. From now on, the cdf $G(x)$ will be referred to as the parent distribution or baseline distribution. The probability density function (pdf) of the new distribution takes the form

$$f(x) = f(x;a,\boldsymbol{\tau}) = \frac{g(x)}{\Gamma(a)}\{-\log[1-G(x)]\}^{a-1}, \quad x \in \mathbb{R}, \tag{1}$$

where $g(x) = g(x;\boldsymbol{\tau}) = \mathrm{d}G(x)/\mathrm{d}x$ is the baseline pdf. For $a = 1$, $f(x) = g(x)$ and, therefore, $g(x)$ is a basic exemplar of (1). Further, if Z has a gamma distribution, $Z \sim \mathrm{Gamma}(a,1)$ say, with density function $h(z) = \Gamma(a)^{-1}z^{a-1}\mathrm{e}^{-z}(z>0)$, then the random variable $X = G^{-1}(1-\mathrm{e}^{-Z})$ has pdf given by Eq. (1). In this paper, we shall refer to (1) as the gamma G (Γ-G) distribution.

Recently, [7] used a similar approach presented in [6] to introduce a new family of distributions generated by gamma random variables. They define $F(x)$ in the form $F(x) = F(x;a,\boldsymbol{\tau}) = 1 - \Gamma(a)^{-1}\gamma(a, -\log[G(x)])$, for $x \in \mathbb{R}$, whereas the pdf is

$$f(x) = f(x;a,\boldsymbol{\tau}) = \frac{g(x)}{\Gamma(a)}\{-\log[G(x)]\}^{a-1}, \quad x \in \mathbb{R}. \tag{2}$$

Some interesting motivations for this new class of distributions are provided by [7]. In particular, if $Z \sim \mathrm{Gamma}(a,1)$, then the random variable $X = G^{-1}(\mathrm{e}^{-Z})$ has pdf given by Eq. (2). Thus, accordingly to [7], the new family of distributions may be regarded as a dual family of the Zografos–Balakrishnan's family of distributions. Also, let Z be a random variable with log-gamma distribution with density function $h(z) = \Gamma(a)^{-1}\exp(az - \mathrm{e}^z)$, $z \in \mathbb{R}$. Then, the random variable $X = G^{-1}(\exp(-\mathrm{e}^Z))$ also has the pdf (2). We shall refer to (2) as the gamma dual G ($\Gamma2$-G) distribution.

The purposes of the present paper are twofold. First, we propose a new representation for the pdf of the Γ-G model as an absolutely convergent power series of the cumulative function of the baseline distribution. Second, we use the generator approach suggested by [6] to define a new model, called the gamma exponentiated Weibull (Γ-EW) distribution, which generalizes the exponentiated exponential, Weibull and EW models. In addition, we investigate some structural properties of the new model and discuss maximum likelihood estimation of its parameters. The proposed model is much more flexible than the Weibull and EW distributions and can be used effectively for modeling positive real data in many areas. A real data example is presented to show the flexibility of the Γ-EW model over other lifetime models in practice.

Recently, a new four-parameter generalization of the Weibull distribution was introduced in [27] by using the generator approach of [7], named here as the *gamma dual exponentiated Weibull distribution* (Γ2-EW). Unfortunately, the expansion for the Γ2-EW density function derived by these authors, which is used to obtain some general properties of this model, is not a valid expansion, i.e. not convergent (see Appendix), and hence some properties of the Γ2-EW distribution presented in their paper like moments, moment generating function, etc., do not work. The general expansion derived in this paper for the Γ-G density function, however, is a valid expansion (i.e. convergent). In particular, we use this general expansion to derive the moments, moment generating function, etc., of the new four-parameter Γ-EW distribution.

2. Expansion for the *G* density function

In what follows, we derive a very useful representation for the Γ-G density function, which can be used to derive general properties (moments, entropy, etc.) of this new class of distributions. It should be noticed that a representation for the $\Gamma2$-G density function can be directly obtained from the representation for the Γ-G density function simply by replacing the baseline cdf $G(x)$ with the survival function of the baseline G distribution, that is, by replacing $G(x)$ with $S(x) = 1 - G(x)$.

It can be shown that

$$\left[-\frac{\log(1-z)}{z}\right]^\delta = 1 + \delta z\sum_{n=0}^\infty \psi_n(n+\delta)z^n, \tag{3}$$

where $\delta \in \mathbb{R}$, $|z| < 1$ and the coefficients $\psi_n(\cdot)$ are Stirling polynomials. These coefficients can be expressed in the form

$$\psi_{n-1}(w) = \frac{(-1)^{n-1}}{(n+1)!}\left[H_n^{n-1} - \frac{w+2}{n+2}H_n^{n-2} + \frac{(w+2)(w+3)}{(n+2)(n+3)}H_n^{n-3} - \cdots + (-1)^{n-1}\frac{(w+2)(w+3)\cdots(w+n)}{(n+2)(n+3)\cdots(2n)}H_n^0\right], \tag{4}$$

where H_n^m are positive integers defined recursively by $H_{n+1}^m = (2n+1-m)H_n^m + (n-m+1)H_n^{m-1}$, with $H_0^0 = 1$, $H_{n+1}^0 = 1 \times 3 \times 5 \times \cdots \times (2n+1)$, and $H_{n+1}^n = 1$. The first six polynomials are $\psi_0(w) = 1/2$, $\psi_1(w) = (2+3w)/24$, $\psi_2(w) = (w+w^2)/48$, $\psi_3(w) = (-8-10w+15w^2+15w^3)/5760$, $\psi_4(w) = (-6w-7w^2+2w^3+3w^4)/11520$ and $\psi_5(w) = (96+140w-224w^2-315w^3+63w^5)/2903040$.

Remark 1. According to another definition,[1] the polynomials $S_0(w) = 1$ and $S_n(w) = n!(w+1)\psi_{n-1}(w)$, $n \geqslant 1$, are also known as *Stirling polynomials*. In this article, we use this

[1] See, for example, http://mathworld.wolfram.com/StirlingPolynomial.html.