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ORIGINAL ARTICLE

On using third and fourth kinds Chebyshev polynomials for solving the integrated forms of high odd-order linear boundary value problems

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Abstract This article presents some spectral Petrov–Galerkin numerical algorithms based on using Chebyshev polynomials of third and fourth kinds for solving the integrated forms of high odd-order two point boundary value problems governed by homogeneous and nonhomogeneous boundary conditions. The principle idea behind obtaining the proposed numerical algorithms is based on constructing trial and test functions as compact combinations of shifted Chebyshev polynomials of third and fourth kinds. The algorithms lead to linear systems with specially structured matrices that can be efficiently inverted. Some numerical examples are illustrated for the sake of demonstrating the validity and the applicability of the proposed algorithms. The presented numerical results indicate that the proposed algorithms are reliable and very efficient.

MATHEMATICS SUBJECT CLASSIFICATION: 65M70; 65N35; 35C10; 42C10

1. Introduction

Chebyshev polynomials have become increasingly crucial in numerical analysis, from both theoretical and practical points of view. It is well-known that there are four kinds of Chebyshev polynomials, and all of them are special cases of the more widest class of Jacobi polynomials. The first and second kinds are special cases of the symmetric Jacobi polynomials (i.e., ultraspherical polynomials), while the third and fourth kinds are special cases of nonsymmetric Jacobi polynomials. In literature, there is a great concentration on first and second kinds of Chebyshev polynomials $T_n(x)$ and $U_n(x)$ and their various uses in numerous applications, (see for instance, [1–3]).

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However, there are few articles concentrate on the other two types of Chebyshev polynomials namely, third and fourth kinds $V_n(x)$ and $W_n(x)$, either from theoretical or practical points of view and their uses in various applications, (see, for example, [4–6]). This motivates our interest in such polynomials. The interested readers in Chebyshev polynomials of third and fourth kinds are refereed to the excellent book of Mason and Handscomb [7].

If we were asked for "a pecking order" of these four Chebyshev polynomials $T_n(x), U_n(x), V_n(x)$ and $W_n(x)$, then we would say that $T_n(x)$ is the most important and versatile. Moreover $T_n(x)$ generally leads to the simplest formulae, whereas results for the other polynomials may involve slight complications. However, all four polynomials have their role. For example, $U_n(x)$ is useful in numerical integration (see, Mason [8]), while $V_n(x)$ and $W_n(x)$ can be useful in situations in which singularities occur at one end point (+1 or -1) but not at the other (see, Mason and Handscomb [7]).

Due to their great importance in several applications, high even-order and high odd-order boundary value problems have been investigated by a large number of authors. Theorems which discuss the conditions for the existence and uniqueness of solutions of such problems are contained in a comprehensive survey in a book by Agarwal [9].

Spectral methods (see, for instance, Boyd [1] and Canuto et al. [10]) are a class of techniques used extensively in applied mathematics and scientific computing to numerically solving ordinary and partial differential equations. The numerical solution is written as an expansion in terms of certain "basis functions", which may be expressed in terms of various orthogonal polynomials. Spectral methods have advantage that they take on a global approach while finite element methods use a local approach, and for this reason, spectral methods have excellent error properties, and converge exponentially.

The study of odd-order equations is of mathematical and physical interest. As an example, third-order equation contains a type of operator which appears in many commonly occurring partial differential equations such as the Kortweg-de Vries equation. Also, fifth-order boundary value problems are of interest as they arise in the mathematical modelling of viscoelastic flows (see, [11,12]). Abd-Elhameed in [13] and Doha and Abd-Elhameed in [14] have constructed efficient spectral-Galerkin algorithms using compact combinations of ultraspherical polynomials for solving the differentiated forms of elliptic equations of high odd-order boundary value problems. Recently, in the two papers of Abd-Elhameed et al. in [15] and Doha et al. in [16], some algorithms for solving numerically the differentiated and integrated forms of third and fifth-order differential equations based on a dual Petrov–Galerkin method using two new families of general parameters generalized Jacobi polynomials, are analyzed.

Of the important high-order differential equations are the singular and singularly perturbed problems (SPPs). These kinds of problems usually appear in quantum mechanics, optimal control, etc. The presence of small parameter in these problems prevents one to obtain satisfactory numerical solutions. It is a well-known fact that the solutions of SPPs have a multi-scale character, that is, there are thin layer(s) where the solution varies very rapidly, while away from the layer(s) the solution behaves regularly and varies slowly. The existence and uniqueness of singularly purturbed boundary value problems was discussed by Howers [17], Kelevedjiev [18], and Roos et al. [19].

As an alternative approach to differentiating solution expansions is to integrate the differential equation q times, where q is the order of the equation. An advantage of this approach is that the resulted algebraic system contains a finite number of terms and hence they are cheaper in solving than those obtained by the differentiated forms. Doha et al. in [16] followed this approach for solving the integrated forms of third- and fifth-order elliptic differential equations. Moreover, Doha and Abd-Elhameed in [4] obtained new formulae for the repeated integrals of Chebyshev polynomials of third and fourth kinds and they used these formulae for solving the integrated forms of sixth-order boundary value problems.

The main objective of the present article is to develop some efficient spectral algorithms based on shifted Chebyshev third and fourth kinds-Galerkin methods for solving the integrated forms of high odd-order differential equations.

The contents of this article are organized as follows. In Section 2, some properties and relations of Chebyshev polynomials of third and fourth kinds and their shifted ones are presented. In Section 3, we discuss some algorithms for solving the integrated forms of high odd-order elliptic differential equations governed by homogeneous and nonhomogeneous boundary conditions using shifted Chebyshev third kind Petrov–Galerkin method (SC3PGM). In Section 4, we are concerned with the same equations but by using shifted Chebyshev fourth kind Petrov–Galerkin method (SC4PGM). Section 5 is concerned with discussing the condition numbers resulted from the application of the two proposed algorithms in Sections 3 and 4. In Section 6, we present three numerical examples including comparisons with some other methods aiming to exhibit the accuracy and the efficiency of our proposed algorithms. Some concluding remarks are presented in Section 7.

2. Some properties of third and fourth kinds of Chebyshev polynomials and their shifted ones

2.1. Chebyshev polynomials of third and fourth kinds

Chebyshev polynomials $V_i(x)$ and $W_i(x)$ of third and fourth kinds are polynomials in x, which can be defined by one of the two following equivalent forms (see, [7]):

$$V_i(x) = \frac{\cos\left(i+\frac{1}{2}\right)\theta}{\cos\frac{\theta}{2}} = \frac{2^{2i}}{\binom{2i}{i}} P_i^{(-\frac{1}{2},\frac{1}{2})}(x),$$

and

$$W_i(x) = \frac{\sin\left(i+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}} = \frac{2^{2i}}{\binom{2i}{i}} P_i^{(\frac{1}{2},-\frac{1}{2})}(x),$$

where $x = \cos\theta$, and $P_i^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree i. It is clear that

$$W_i(x) = (-1)^i\ V_i(-x). \tag{1}$$

The polynomials $V_i(x)$ and $W_i(x)$ are orthogonal on $(-1,1)$ with respect to the weight functions $\sqrt{\frac{1+x}{1-x}}$ and $\sqrt{\frac{1-x}{1+x}}$, respectively, i.e.,