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SHORT COMMUNICATION

Refined Young inequalities with Specht's ratio

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Abstract In this paper, we show that the v-weighted arithmetic mean is greater than the product of the v-weighted geometric mean and Specht's ratio. As a corollary, we also show that the v-weighted geometric mean is greater than the product of the v-weighted harmonic mean and Specht's ratio. These results give the improvements for the classical Young inequalities, since Specht's ratio is generally greater than 1. In addition, we give an operator inequality for positive operators, applying our refined Young inequality.

1. Introduction

We start from the famous Young inequality:

$$(1-v)a + vb \geqslant a^{1-v}b^{v} \tag{1}$$

for positive numbers a, b and $v \in [0,1]$. The inequality (1) is also called v-weighted arithmetic-geometric mean inequality and its reverse inequality was given in [1] with Specht's ratio as follows:

$$S\left(\frac{a}{b}\right)a^{1-v}b^{v} \geqslant (1-v)a + vb \tag{2}$$

for positive numbers a, b and $v \in [0,1]$, where the Specht's ratio [2,3] was defined by

$$S(h) \equiv \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}, \quad (h \neq 1)$$

for a positive real number h.

Recently, based on the refined Young inequality [4,5]:

$$(1-v)a + vb \geqslant a^{1-v}b^{v} + r\left(\sqrt{a} - \sqrt{b}\right)^{2}, \tag{3}$$

for positive numbers a, b and $v \in [0,1]$, where $r \equiv \min\{v, 1-v\}$, we proved the following operator inequalities:

Proposition 1 [6]. *For $v \in [0,1]$ and positive operators A and B, we have*

$$\begin{aligned}(1-v)A + vB &\geqslant A\sharp_{v}B + 2r\left(\frac{A+B}{2} - A\sharp_{1/2}B\right)\\ &\geqslant A\sharp_{v}B\\ &\geqslant \left\{A^{-1}\sharp_{v}B^{-1} + 2r\left(\frac{A^{-1}+B^{-1}}{2} - A^{-1}\sharp_{1/2}B^{-1}\right)\right\}^{-1}\\ &\geqslant \left\{(1-v)A^{-1} + vB^{-1}\right\}^{-1}\end{aligned}$$

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where $r \equiv \min\{v, 1-v\}$ *and* $A\sharp_v B \equiv A^{1/2}(A^{-1/2}BA^{-1/2})^v A^{1/2}$ *defined for* $v \in [0,1]$.

The above inequalities can be regarded as an additive-type refinement for the Young inequalities [7,8]:

$$(1-v)A + vB \geqslant A\sharp_v B \geqslant \left\{(1-v)A^{-1} + vB^{-1}\right\}^{-1}. \quad (4)$$

In this short paper, we give a multiplicative-type refinement for the Young inequalities (4) with the Specht's ratio.

2. Main results

We here review the properties of the Specht's ratio. See [1–3] for example, as for the proof and the details.

Lemma 1. *The Specht's ratio*

$$S(h) \equiv \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}, \quad (h \neq 1,\ h > 0)$$

has the following properties.

(i) $S(1) = 1$ *and* $S(h) = S(1/h) > 1$ *for* $h > 0$.
(ii) $S(h)$ *is a monotone increasing function on* $(1,\infty)$.
(iii) $S(h)$ *is a monotone decreasing function on* $(0,1)$.

We use the following lemmas to show our theorem.

Lemma 2. *For* $x \geqslant 1$, *we have*

$$\frac{2(x-1)}{x+1} \leqslant \log x \leqslant \frac{x-1}{\sqrt{x}}. \quad (5)$$

Proof 1. We firstly prove the second inequality of (5). We put $\sqrt{x} = t$ and

$$f(t) \equiv \frac{t^2-1}{t} - 2\log t, \quad (t \geqslant 1).$$

Then we have $f'(t) = \left(\frac{t-1}{t}\right)^2 \geqslant 0$ and $f(1) = 0$. Thus we have $f(t) \geqslant f(1) = 0$ and then we have $\log t^2 \leqslant \frac{t^2-1}{t}$, which implies the second inequality in (5).

We also put

$$g(x) \equiv (x+1)\log x - 2(x-1), \quad (x \geqslant 1).$$

Then we have $g(1) = 0$, $g'(x) = \log x + \frac{x+1}{x} - 2$, $g'(1) = 0$ and $g''(x) = \frac{x-1}{x^2} \geqslant 0$. Therefore we have $g(x) \geqslant g(1) = 0$, which implies the first inequality in (5). □

Note that Lemma 2 can be also proven by the following relation for three means:

$$\sqrt{xy} < \frac{x-y}{\log x - \log y} < \frac{x+y}{2}$$

for positive real numbers x and y, where $x \neq y$.

Lemma 3. *For* $t > 0$, *we have*

$$e(t^2+1) \geqslant (t+1)t^{\frac{t}{t-1}}. \quad (6)$$

Proof 2. We firstly prove the inequality (6) for $t \geqslant 1$. We put

$$f(t) = e(t^2+1) - (t+1)t^{\frac{t}{t-1}}.$$

By using the first inequality of (5), we have

$$\begin{aligned} f'(t) &= \frac{2t(t-1)^2 e + 2t(1-t)t^{\frac{t}{t-1}} + t^{\frac{t}{t-1}}(t+1)\log t}{(t-1)^2} \\ &\geqslant \frac{2t(t-1)^2 e + 2t(1-t)t^{\frac{t}{t-1}} + 2(t-1)t^{\frac{t}{t-1}}}{(t-1)^2} \\ &= \frac{2t(t-1)^2 e - 2t(t-1)^2 t^{\frac{1}{t-1}}}{(t-1)^2} \\ &\geqslant \frac{2t(t-1)^2 t^{\frac{1}{t-1}} - 2t(t-1)^2 t^{\frac{1}{t-1}}}{(t-1)^2} = 0. \end{aligned}$$

In the last inequality, we have used the fact that $\lim_{t\to 1} t^{\frac{1}{t-1}} = e$ and the function $t^{\frac{1}{t-1}}$ is monotone decreasing on $t \in [1,\infty)$. We also have $f(1) = 0$ so that we have $f(t) \geqslant 0$ which proves the following inequality:

$$e(t^2+1) \geqslant (t+1)t^{\frac{t}{t-1}}, \quad t \geqslant 1.$$

Putting $t = \frac{1}{s}$ in the above inequality with simple calculations, we have

$$e(s^2+1) \geqslant (s+1)s^{\frac{s}{s-1}}, \quad 0 < s \leqslant 1. \qquad \square$$

Then we have the following inequality which improves the classical Young inequality between v-weighted geometric mean and v-weighted arithmetic mean.

Theorem 1. *For* $a,b > 0$ *and* $v \in [0,1]$,

$$(1-v)a + vb \geqslant S\left(\left(\frac{b}{a}\right)^r\right)a^{1-v}b^v, \quad (7)$$

where $r \equiv \min\{v, 1-v\}$ *ans* $S(\cdot)$ *is the Specht's ratio.*

Proof 3. We prove the following inequality

$$\frac{(b-1)v+1}{b^v S(b^v)} = \frac{e\{(b-1)v+1\}\log b^v}{(b^v)^{\frac{b^v}{b^v-1}}(b^v-1)} \geqslant 1 \quad (8)$$

in the case of $0 \leqslant v \leqslant \frac{1}{2}$. From Lemma 2, we have

$$\frac{\log b^v}{b^v - 1} \geqslant \frac{2}{b^v+1}, \quad b > 0.$$

Therefore we have the following first inequality:

$$\frac{e\{(b-1)v+1\}\log b^v}{(b^v)^{\frac{b^v}{b^v-1}}(b^v-1)} \geqslant \frac{2e\{(b-1)v+1\}}{(b^v)^{\frac{b^v}{b^v-1}}(b^v+1)} \geqslant 1, \quad (9)$$

thus we have only to prove the above second inequality. For this purpose, we put the following function f_b on $v \in \left[0, \frac{1}{2}\right]$ for $b > 0$:

$$f_b(v) \equiv 2e\{(b-1)v+1\} - (b^v)^{\frac{b^v}{b^v-1}}(b^v+1).$$

Then we have

$$\begin{aligned} f_b''(v) = &-\frac{(\log b)^2}{(b^v-1)^4}(b^v)^{\frac{2b^v-1}{b^v-1}}\Big\{(b^v-1)^2(4b^{2v} - 5b^v - 1) \\ &- \left(b^v-1\right)^2(3b^v+1)\log b^v + b^v(b^v+1)(\log b^v)^2\Big\}. \end{aligned}$$

For the case of $b \geqslant 1$, using the inequalities (5), we have