Journal of the Egyptian Mathematical Society (2016) **24**, 165–170

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

Original Article

Picard, Adomian and predictor–corrector methods for an initial value problem of arbitrary (fractional) orders differential equation

Adel S. Mohamed, R.A. Mahmoud*

Zagazig University, Faculty of Science, Egypt

Received 24 November 2014; revised 27 December 2014; accepted 1 January 2015
Available online 10 February 2015

Keywords

Fractional differential equation;
Picard method;
Adomian method;
Predictor corrector method;
Convergence analysis;
Error analysis

Abstract We study the two analytical methods, the classical method of successive approximations (Picard method), Adomian decomposition method (ADM) see (Abbaoui and Cherruault, 1994; Adomian et al., 1992; Adomian, 1995) [1–3] and the (numerical method) predictor corrector method (PECE) for an initial value problem of arbitrary (fractional) orders differential equation (FDE). The existence and uniqueness of the solution will be proved and the convergence will be discussed for each method. Some examples will be studied.

2000 Mathematical Subject Classification: 26A33; 26A18; 39B12

1. Introduction

Let $\alpha \in [0, 1)$. In this paper, we study the existence and uniqueness of the solution of the initial value problem

$$\frac{dx}{dt} + D^{\alpha} x\ (t) = f(t, x(t)), \quad 0 < \alpha < 1, \tag{1}$$

$$x(0) = \tilde{x}. \tag{2}$$

We apply the three methods Adomian, Picard and predictor–corrector to obtain numerical solution of the problem (1) and (2).

Now, the definition of the fractional-order integral and differential operators are given by the following.

Definition 1. Let β be a positive real number, the fractional-order integral of order β of th function f is defined on the interval $[0, T]$ by

$$I_0^{\beta} f(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s) ds$$

and the fractional-order derivative of the function $f \in C^1[0, T]$ of order $\alpha \in (0, 1]$ is defined by

$$D^{\alpha} f(t) = I^{1-\alpha} \frac{df}{dt}.$$

* Corresponding author.
E-mail addresses: 3adel@live.nl (A.S. Mohamed), rony_695@yahoo.com (R.A. Mahmoud).

Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.joems.2015.01.001

2. Uniqueness theorem

Now, the initial value problem (1) and (2) will be investigated under the following assumptions:

(i) $f : J = [0, T] \times D \to R$ is continuous where D is a closed subset of R;
(ii) f satisfies the Lipschitz condition with Lipschitz constant L
i.e

$$|f(t,x) - f(t,y)| \le L|x-y|, \quad \forall (t,x), (t,y) \in I \times D.$$

Let C = C(J) be the space of all real-valued functions which are continuous on J.

Definition 2. By a solution of the problem (1) and (2) we mean a function $x \in C[0, T]$. This function satisfies the problem (1) and (2).

Let $x(t)$ be a solution of the initial value problem (1) and (2). Integrating (1) we obtain

$$x(t) - \tilde{x} + I^{1-\alpha}\big(x(t) - \tilde{x}\big) = I f(t,x),$$

then we have

$$x(t) = \tilde{x}\left(1 + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right) - \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)ds + \int_0^t f(s, x(s))ds. \quad (3)$$

Now let $x \in C(J)$ be a solution of the integral Eq. (3), then

$$\frac{dx}{dt} = 0 + \tilde{x}\frac{(1-\alpha)t^{-\alpha}}{\Gamma(2-\alpha)} - \frac{d}{dt}\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)ds + \frac{d}{dt}\int_0^t f(s,x(s))ds = \tilde{x}\frac{t^{-\alpha}}{\Gamma(1-\alpha)} - \tilde{x}\frac{t^{-\alpha}}{\Gamma(1-\alpha)} - I^{1-\alpha}\frac{dx}{dt} + f(t,x)$$

and

$$\frac{dx}{dt} + D^{\alpha}x(t) = f(t, x(t))$$

also

$$\tilde{x} = \tilde{x}\left(1 + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right)_{t=0} - \left(\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)ds\right)_{t=0} + \left(\int_0^t f(s,x(s))ds\right)_{t=0}$$

then the problem (1) and (2) and the integral Eq. (3) are equivalent.

Comparison between analytical methods is studied in many papers, for examples [4–7].

Define the operator F as

$$(Fx)(t) = \tilde{x}\left(1 + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right) - \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)ds + \int_0^t f(s,x(s))ds, \quad \alpha > 0, \ \forall x \in C.$$

Theorem 1. *Let the assumptions (i)–(ii) be satisfied if* $LT + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} < 1$ *, then the initial value problem* (1) *has a unique solution* $x \in C$.

Proof. Firstly we prove that $F : C \to C$ is continuous.
let $x \in C(J)$, t_1, $t_2 \in J$ such that $|t_2 - t_1| < \delta$

$$Fx(t_2) - Fx(t_1) = \tilde{x}\frac{t_2^{1-\alpha}}{\Gamma(2-\alpha)} - \int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)ds + \int_0^{t_2} f(s,x(s))ds - \tilde{x}\frac{t_1^{1-\alpha}}{\Gamma(2-\alpha)} + \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)ds - \int_0^{t_1} f(s,x(s))ds$$

$$|Fx(t_2) - Fx(t_1)| \le |\tilde{x}|\frac{\left|t_2^{1-\alpha} - t_1^{1-\alpha}\right|}{\Gamma(2-\alpha)} - \int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)}|x(s)|ds + \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)}|x(s)|ds + \int_{t_1}^{t_2} |f(s,x(s))|ds$$

$$\begin{aligned}\|Fx(t_2) - Fx(t_1)\| &= \max_{t\in J}|Fx(t_2) - Fx(t_1)| \\ &\le \|\tilde{x}\|\frac{\left|t_2^{1-\alpha} - t_1^{1-\alpha}\right|}{\Gamma(2-\alpha)} - \|x\|\int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)}ds + \|x\|\int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)}ds + k\int_{t_1}^{t_2} ds \\ &\le \|\tilde{x}\|\frac{\left|t_2^{1-\alpha} - t_1^{1-\alpha}\right|}{\Gamma(2-\alpha)} + \frac{\|x\|}{\Gamma(2-\alpha)}\left|t_2^{1-\alpha} - t_1^{1-\alpha}\right| + k|t_2 - t_1| \\ &\le \frac{\left(\|\tilde{x}\| + \|x\|\right)}{\Gamma(2-\alpha)}\left|t_2^{1-\alpha} - t_1^{1-\alpha}\right| + k|t_2 - t_1| \le \epsilon\end{aligned}$$

where

$$|f(t, x(t))| \le k,$$

This proves that $F : C[0, T] \to C[0, T]$.

Now we prove that F is contraction, for this we have

$$Fx - Fy = -\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s)ds + \int_0^t f(s,x(s))ds + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} y(s)ds - \int_0^t f(s,y(s))ds$$

$$\begin{aligned}|Fx - Fy| &\le \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}|x(s) - y(s)|ds + \int_0^t |f(s,x(s)) - f(s,y(s))|ds \\ &\le \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}|x(s) - y(s)|ds + L\int_0^t |x(s) - y(s)|ds \\ &\le \|x-y\|\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}ds + LT\|x-y\|\end{aligned}$$

$$\|Fx - Fy\| \le \|x-y\|\left(LT + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}\right).$$