Journal of the Egyptian Mathematical Society (2016) **24**, 181–186

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

Original Article

Applications of the differential operator to a class of meromorphic univalent functions

Khalida Inayat Noor[a], Qazi Zahoor Ahmad[a,1], Janusz Sokół[b,*]

[a] *Department of Mathematics, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan*
[b] *Department of Mathematics, Rzeszów University of Technology, Al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland*

Received 19 February 2015; accepted 4 April 2015
Available online 16 May 2015

Keywords

Meromorphic functions;
Close-to-convex functions;
Convolution;
Differential operator;
Inclusion results

Abstract In this paper, we define a new subclass of meromorphic close-to-convex univalent functions defined in the punctured open unit disc by using a differential operator. Some inclusion results, convolution properties and several other properties of this class are studied.

1991 Mathematics Subject Classification: 30C45; 30C50

1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and univalent in $E^* = \{z : 0 < |z| < 1\} = E\{0\}$. For the functions

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k \text{ and } g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k,\ z \in E^*,$$

analytic in E^*, their Hadamard product or convolution, $f * g$, is the function defined by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k b_k z^k, \quad z \in E^*,$$

where $(*)$ stands for convolution sign.

The theory of linear operators plays an important role in geometric function theory. Several differential and integral operators were introduced and studied, see for example [1,3,16,21,22,25,27]. For the recent work on linear operators for

* Corresponding author.
E-mail addresses: khalidanoor@hotmail.com (K.I. Noor), zahoorqazi5@gmail.com (Q.Z. Ahmad), jsokol@prz.edu.pl (J. Sokół).
1 Current address.
Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

S1110-256X(15)00028-0
http://dx.doi.org/10.1016/j.joems.2015.04.001

meromorphic functions, we refer to [4,6,10,11]. In this work we consider the operator defined by El-Ashwah [10] and El-Ashwah and Aouf [11,12]. For λreal, $l > 0$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the linear operator $D^n(\lambda,l) : \Sigma \to \Sigma$ was defined by

$$D^n f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left[\frac{l+\lambda(k+1)}{l}\right]^n a_k z^k, \quad z \in E^*. \tag{1.2}$$

Clearly $D^0 f(z) = f(z)$ and $D^1(1,1)f(z) = 2f(z)) + zf'(z)$.

It is noted that

$$\lambda z\big(D^n f(z)\big)'^{n+1} f(z) - (\lambda + l)D^n f(z), \quad z \in E^*. \tag{1.3}$$

For $\lambda = 1$, the operator $D^n(1,l)f(z)$ was introduced and studied by Cho et al. [7,8]. The case $D^n(\lambda,1)f(z)$ was considered by Al-Oboudi and Al-Zkeri [2]. Further the operators $D^n(1,1)f(z)$ and $D^1(-1,1)f(z)$ were investigated by Uralegaddi and Somanatha [27] and Noor and Ahmad [23] respectively.

For $\alpha,(0 \le \alpha < 1)$, a function $f(z) \in \Sigma$ is said to be meromorphic starlike and convex of order α if it satisfies

$$-\Re e\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in E,$$

and

$$-\Re e\left\{\frac{\big(zf'(z)\big)'}{f'(z)}\right\} > \alpha, \quad z \in E,$$

respectively. We denote the former class of functions as $\Sigma^*(\alpha)$ and the later one by $\Sigma^k(\alpha)$. These classes have been studied by Pommerenke [24], Clunie [9] and Miller [19,20]. Further a function $f(z) \in \Sigma$ is said to be from the class $\Sigma^c(\alpha)$, if it satisfies

$$-\Re e\{z^2 f'(z)\} > \alpha, \quad z \in E. \tag{1.4}$$

This class was investigated by Ganigi and Uralegaddi [14], Cho and Owa [5] and Wang and Guo [28].

Definition 1. A function f given by (1.1) is said to belong to the class $\Sigma^g(\alpha)$ of meromorphic close-to-convex functions if there exists a function $g \in \Sigma^*(\alpha)$ such that

$$-\Re e\left\{\frac{zf'(z)}{g(z)}\right\} > 0, \quad z \in E.$$

This class of functions was introduced and studied by Libera and Robertson [17].

Remark 1. In [14] it was shown that if a function $f(z) \in \Sigma^c(\alpha)$, then it is meromorphic close-to-convex of order α.

Let f and g be two analytic functions in E. We say that f is subordinate to g, written $f \prec g$, if there exists a Schwarz function $w(z)$, analytic in E with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. If g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

A sequence of non-negative numbers $\{c_n\}$ is said to be a convex null sequence if $c_k \to 0$ as $k \to \infty$ and

$$c_0 - c_1 \ge c_1 - c_2 \ge \cdots \ge c_k - c_{k+1} \ge \cdots \ge 0.$$

Now we define the following class of functions by using the operator defined in (1.2).

Definition 2. A function $f(z) \in \Sigma$ is said to be in the class $\Sigma^n(\lambda,\alpha)$, if and only if

$$-\Re e\left\{z^2\big(D^n f(z)\big)'\right\} > \alpha, \quad z \in E, \quad (n \in \mathbb{N}_0).$$

When $n = 0$, we obtain the class $\Sigma^c(\alpha)$ of meromorphic functions, which was studied by Ganigi and Uralegaddi [14], Cho and Owa [5] and Wang and Guo [28].

2. Preliminary results

We need the following results.

Lemma 1 [26]. *If $p(z)$ is analytic in E with $p(0) = 1$ and $\Re e\{p(z)\} > 1/2, z \in E$, then for any analytic function F, in E, the function $P * F$ takes its values in the convex hull of $F(E)$.*

Lemma 2 [13]**.** *Let $\{c_k\}_{k=0}^{\infty}$ be a convex null sequence. Then the function*

$$p(z) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k z^k, \quad z \in E,$$

is analytic and $\Re e\{p(z)\} > 0$ in E.

The following result is due to Hallenbeck and Ruscheweyh.

Lemma 3 [15]**.** *Let the function $h(z)$ be convex univalent in E with*

$$h(0) = 1, \quad \gamma \ne 0 \quad \text{and} \quad \Re e\gamma > 0, \quad z \in E.$$

Suppose that the function

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots,$$

is analytic in E and satisfying the following differential subordination

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z), \quad z \in E,$$

then

$$p(z) \prec q(z) \prec h(z), \quad z \in E,$$

where

$$q(z) = \frac{\gamma}{z^\gamma}\int_0^z h(t)t^{\gamma-1}dt.$$

The function $q(z)$ is convex and is the best dominant.

Lemma 4 [18]**.** *Let $q(z)$ be a convex function in E and let*

$$h(z) = q(z) + \beta zq'(z),$$

where $\beta > 0$. If $p(z)$ is analytic and satisfies

$$p(z) + \beta zp'(z) \prec h(z), \quad z \in E,$$